The Brauer loop scheme with boundaries

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qKZ - Background

Quantum Knizhnik–Zamolodchikov equation

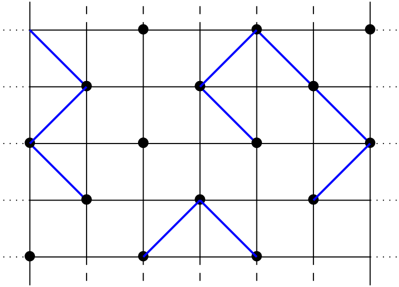
$$R_i(z_i - z_{i+1})|\Psi(z_1, \dots, z_i, z_{i+1}, \dots, z_L)\rangle = |\Psi(z_1, \dots, z_{i+1}, z_i, \dots, z_L)\rangle \quad \forall i$$

$+$ boundary relations

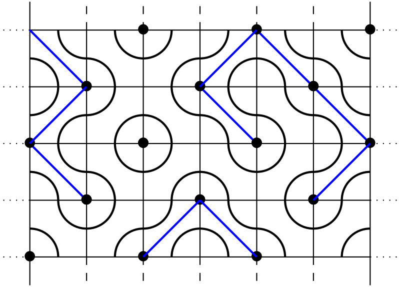
LHS acts on basis vectors, RHS on coefficients

Arises in lattice models - from interlacing condition of transfer matrix

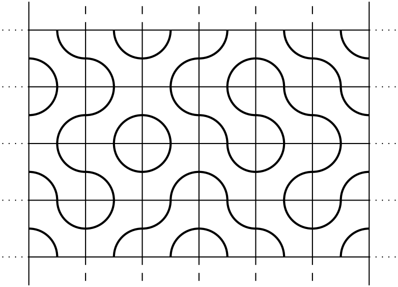
Loop models - Percolation



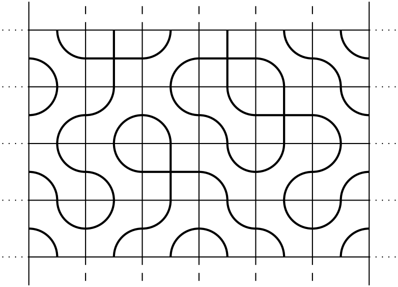
Loop models - Temperley–Lieb



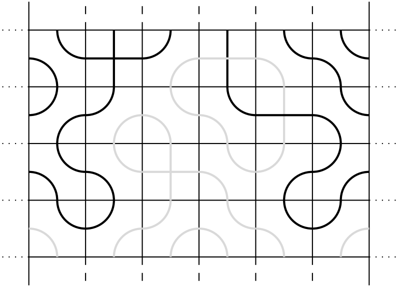
Loop models - Temperley–Lieb



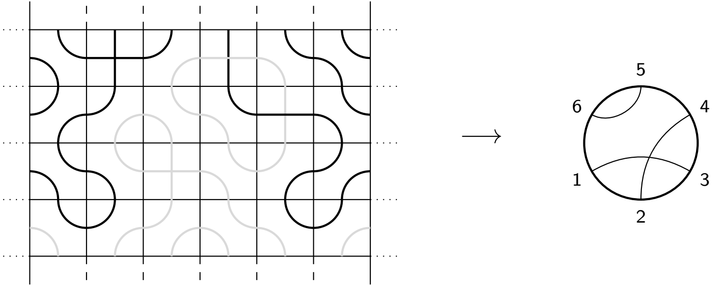
Loop models - Brauer



Crossing link patterns



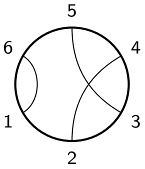
Crossing link patterns



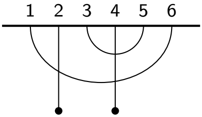
Crossing link patterns

LP_L^a is the set of all crossing link patterns with boundary condition a:

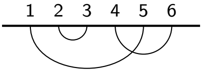
Periodic



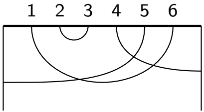
Identified



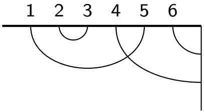
Closed



Open

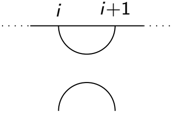


Mixed

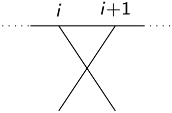


Brauer algebra – bulk

$e_i :=$



$f_i :=$



satisfying

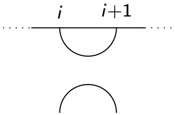
$$e_i^2 = \beta\ e_i \qquad f_i e_i = e_i$$

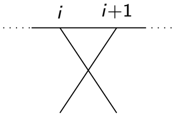
$$f_i^2 = 1 \qquad e_i f_i = e_i$$

$$e_i e_{i\pm1} e_i = e_i \qquad f_i e_{i\pm1} e_i = f_{i\pm1} e_i$$

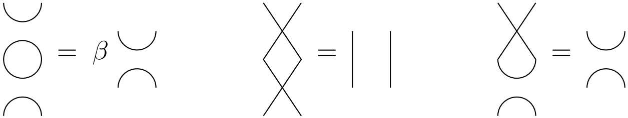
$$f_i f_{i+1} f_i = f_{i+1} f_i f_{i+1} \qquad e_i e_{i\pm1} f_i = e_i f_{i\pm1}$$

closed loop weight $\beta = \dfrac{A - \epsilon}{A - \epsilon/2}$

$e_i :=$ 

$f_i :=$ 

satisfying



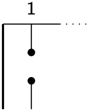
closed loop weight $\beta = \dfrac{A - \epsilon}{A - \epsilon/2}$

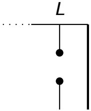
Brauer algebra – boundary conditions

Periodic:

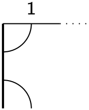
$$L + 1 \equiv 1$$

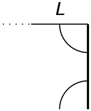
Identified:

$e_0 =$ 

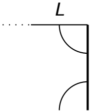
$e_L =$ 

Open:

$e_0 =$ 

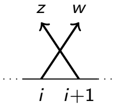
$e_L =$ 

Mixed:

$e_L =$ 

R-matrix

$$\check{R}_i(z-w) = a(z-w) + b(z-w)\ e_i + c(z-w)\ f_i \ =$$

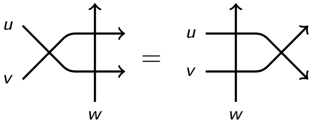


Coefficients:

$$a(z) = \frac{(2A-\epsilon)(A-z)}{r(z)} \quad b(z) = \frac{(2A-\epsilon)z}{r(z)} \quad c(z) = \frac{z(A-z)}{r(z)}$$

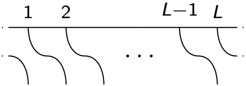
where $r(z) = (A+z)(2A-z-\epsilon)$.

Chosen so that Yang–Baxter equation holds:



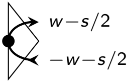
Boundary operators

Periodic case: Rotation operator

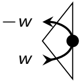
$\sigma =$ 

Rest: K-matrices (Identity if trivial boundary)

$$\check{K}_0(w) = \frac{(A-2w)}{(A+2w)} + \frac{4w}{(A+2w)}\ e_0 \ =$$



$$\check{K}_L(w) = \frac{(A-2w)}{(A+2w)} + \frac{4w}{(A+2w)}\ e_L \ =$$



Chosen so that boundary YBE holds

Brauer qKZ equation (local relations)

$$|\Psi(z_1,\ldots,z_L)\rangle \in \mathrm{span}(\mathrm{LP}_L)$$

Exchange relation:

$$\check{R}_i(z_i - z_{i+1})|\Psi(z_1,\ldots,z_i,z_{i+1},\ldots,z_L)\rangle = |\Psi(z_1,\ldots,z_{i+1},z_i,\ldots,z_L)\rangle$$

- Rotation equation (periodic):

$$\sigma|\Psi(z_1,\ldots,z_L)\rangle = |\Psi(z_2,\ldots,z_L,z_1+s)\rangle$$

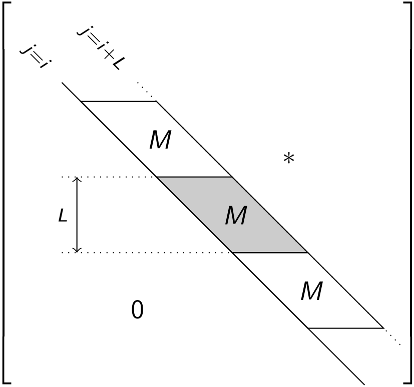
- Boundary exchange relations (rest):

$$\check{K}_0(-z_1 - s/2)|\Psi(z_1,\ldots,z_L)\rangle = |\Psi(-z_1 - s, z_2,\ldots,z_L)\rangle$$
$$\check{K}_L(z_L)|\Psi(z_1,\ldots,z_L)\rangle = |\Psi(z_1,\ldots,z_{L-1},-z_L)\rangle$$

$\to$ Looking for a solution $|\Psi\rangle$ with *polynomial* entries

The Brauer loop scheme (periodic)

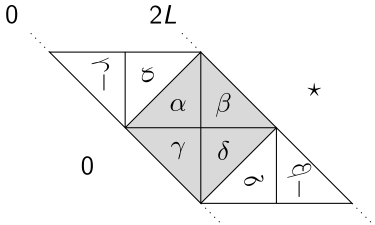
$\mathcal{M}_L^{\mathrm{p}} =$ Infinite upper-triangular (L, L)-periodic strip matrices:



$$E_L^{\mathrm{p}} = \left\{ M \in \mathcal{M}_L^{\mathrm{p}} \;\middle|\; (M^2)_{ij} = 0 \text{ for } i \le j < i + L \right\}$$

The Brauer loop scheme (identified boundaries)

$$E^{\mathrm{i}}_{2L} = E^{\mathrm{p}}_{2L} \cap \left\{ M \in \mathcal{M}^{\mathrm{p}}_{2L} \;\middle|\; M = JM^T J \right\}$$

$M =$ 

The Brauer loop scheme

Claims:

1. The irreducible components (varieties) of E_N^a are labelled in a prescribed way by link patterns, E_α^a.
2. The multidegrees of the varieties satisfy the Brauer qKZ equation.

Proofs: Knutson & Zinn-Justin 2007 (periodic), A.P. & Zinn-Justin 2014 (identified, closed, mixed, open)

$\epsilon \to 0$: Solution of qKZ is ground state of Brauer loop model.

Claim 1: Example $L = 3$, periodic

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} & * & * & * \\ 0 & m_{22} & m_{23} & m_{21} & * & * \\ 0 & 0 & m_{33} & m_{31} & m_{32} & * \end{pmatrix}$$

$E_L^{\mathrm{p}} : (M^2 = 0)$

$$\begin{cases} m_{11}^2 = 0, & m_{11}m_{12} + m_{12}m_{22} = 0, & m_{11}m_{13} + m_{12}m_{23} + m_{13}m_{33} = 0 \\ m_{22}^2 = 0, & m_{22}m_{23} + m_{23}m_{33} = 0, & m_{22}m_{21} + m_{23}m_{31} + m_{21}m_{11} = 0 \\ m_{33}^2 = 0, & m_{33}m_{31} + m_{31}m_{11} = 0, & m_{33}m_{32} + m_{31}m_{12} + m_{32}m_{22} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} m_{11} = 0, & m_{12}m_{23} = 0 \\ m_{22} = 0, & m_{23}m_{31} = 0 \\ m_{33} = 0, & m_{31}m_{12} = 0 \end{cases}$$

$\Rightarrow$ Irreducible components:

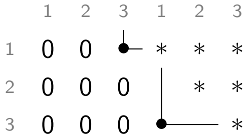
$$\{M \in E_L^{\mathrm{p}} \mid m_{ii} = m_{12} = m_{23} = 0\}$$
$$\{M \in E_L^{\mathrm{p}} \mid m_{ii} = m_{12} = m_{31} = 0\}$$
$$\{M \in E_L^{\mathrm{p}} \mid m_{ii} = m_{23} = m_{31} = 0\}$$

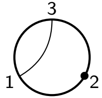
Claim 1: Example $L = 3$, periodic

Correspondence with link pattern:

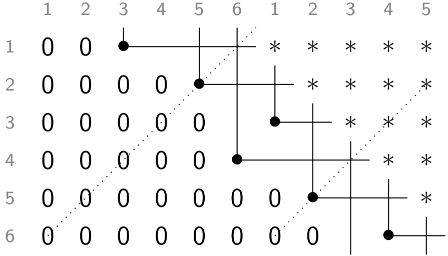
$$E_\pi^{\mathrm{p}} = \{M \in E_L^{\mathrm{p}} \mid m_{ii} = m_{12} = m_{23} = 0\}$$

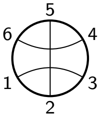
$$\begin{pmatrix} 0 & 0 & m_{13} & * & * & * \\ 0 & 0 & 0 & m_{21} & * & * \\ 0 & 0 & 0 & m_{31} & m_{32} & * \end{pmatrix} \quad \rightarrow$$

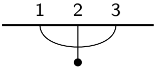


$\pi =$ 

$$\begin{array}{ccccccccccc}
0 & 0 & m_{13} & m_{14} & m_{15} & m_{16} & * & * & * & * & * \\
0 & 0 & 0 & 0 & m_{25} & m_{15} & m_{21} & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & m_{14} & m_{31} & m_{32} & * & * & * \\
0 & 0 & 0 & 0 & 0 & -m_{13} & m_{41} & m_{42} & m_{43} & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & m_{52} & m_{42} & -m_{32} & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m_{41} & -m_{31} & -m_{21}
\end{array}$$







Claim 2: Multidegrees satisfy qKZ

Multidegree: Polynomial associated to a scheme in “a prescribed way”.

$$\operatorname{mdeg} E_L = \sum_{\pi} \operatorname{mdeg} E_\pi$$

Example: The scheme

$$E = \{\ M \in \mathcal{M} \mid M_{ij} = 0 \text{ for some particular } i, j\ \}$$

has multidegree in $\mathcal{M}$

$$\operatorname{mdeg}_{\mathcal{M}} E = (A + z_i - z_j)$$

Generalization of degree of scheme (related to homogeneous degree of defining equations).

Claim 2: Multidegrees satisfy qKZ

Multidegrees of irred. components of Brauer scheme written as a vector

$$|\Phi(z_1,\ldots,z_L)\rangle = \sum_{\pi} \mathrm{mdeg}_{\mathcal{M}}\, E_\pi |\pi\rangle$$

satisfies the Brauer qKZ equation

$$R_i(z_i - z_{i+1})|\Phi(z_1,\ldots,z_i,z_{i+1},\ldots,z_L)\rangle = |\Phi(z_1,\ldots,z_{i+1},z_i,\ldots,z_L)\rangle \quad \forall i$$

$+$ boundary relations

Actions on irred. components (intersection, orbit) give unions of irred. components

$\rightarrow$ qKZ equation on multidegrees

Conclusion

- Allows rigorous proof of solution to qKZ
- More importantly, ∃ unexplained connection between algebraic schemes and integrable models

qKZ equation with BCs	$\leftrightarrow$	Brauer loop scheme with symmetries
Link patterns	$\leftrightarrow$	Irreducible components
Solution components	$\leftrightarrow$	Multidegrees of irreducible components
Solution vector normalization	$\leftrightarrow$	Multidegree of scheme
Integrable Model	$\leftrightarrow$	**Algebraic Variety**

Thank you for your attention

arXiv:1410.0262