A Solution of the q-deformed Knizhnik-Zamolodchikov Equation

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Part 1:
Link patterns and the 2 boundary Temperley-Lieb algebra (2BTL)

2BTL - A Link Pattern Representation

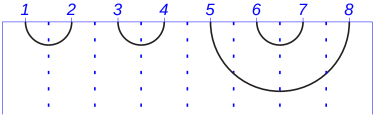
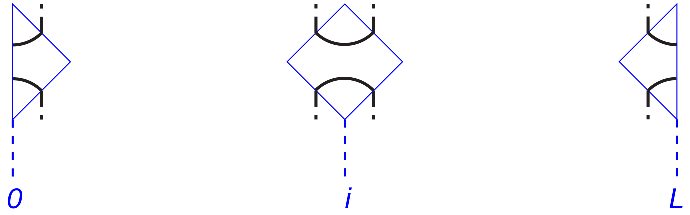


Figure: $|\alpha\rangle$, link pattern of size 8

Shorthand notation: ()()(()).

There are 2^L link patterns in a system of size L.

Temperley-Lieb generators



e_0 e_i e_L

Action of generators

$$\{e_i;\ i = 0, \ldots, L\}$$

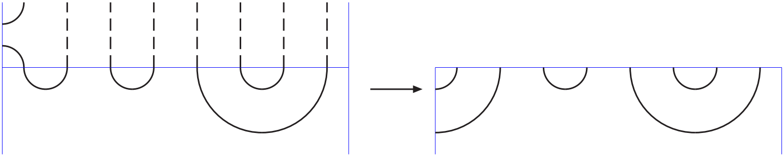


Figure: $e_0|\alpha\rangle$

e_0 acts between the left boundary and position 1

Action of generators

$$\{e_i;\ i = 0, \ldots, L\}$$

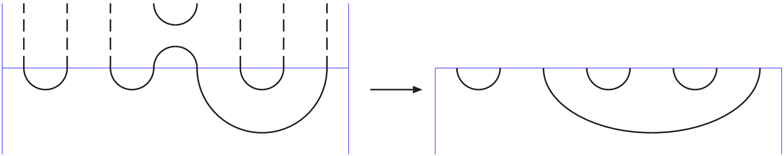


Figure: $e_4|\alpha\rangle$

e_i acts between position i and position $i+1$

Action of generators

$$\{e_i;\ i = 0, \ldots, L\}$$

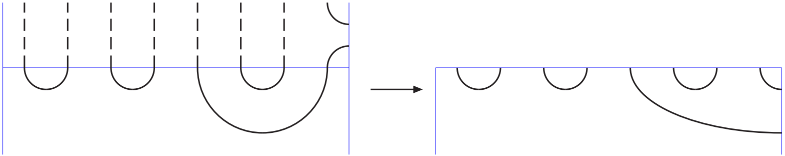
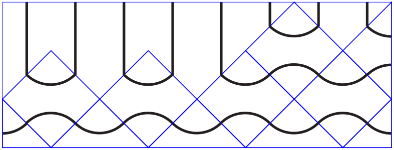


Figure: $e_8|\alpha\rangle$

e_L acts between position L and the right boundary

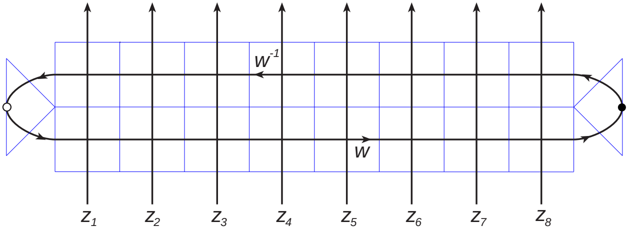
Link pattern



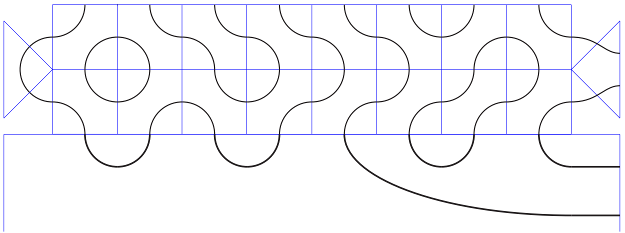
Shorthand notation:)()))(())

Part 2:
Transfer Matrix

Transfer Matrix



Transfer Matrix



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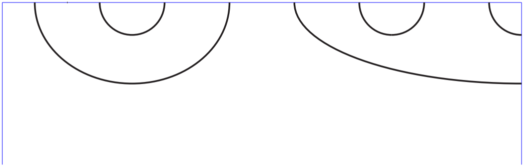
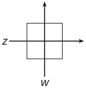
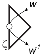


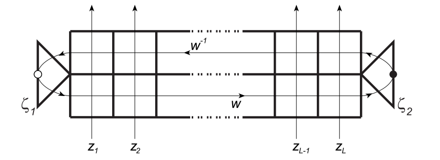
Transfer Matrix

$R(z,w) = a(z/w)$  $+\, b(z/w)$  $=$ 

$K_0(w,\zeta) = c(qw,\zeta)$  $+\, d(qw,\zeta)$  $=$ 

$K_L(w,\zeta) = c(w,\zeta)$  $+\, d(w,\zeta)$  $=$ 

Transfer matrix

$T_L(w; z_1, \ldots, z_L) =$ 

Transfer Matrix

- T_L has a unique eigenvector $|\Psi\rangle = \sum_\alpha \psi_\alpha(z_1, \ldots, z_L)|\alpha\rangle$ with eigenvalue 1

$$T_L|\Psi_L\rangle = |\Psi_L\rangle$$

- The ψ_α are polynomials in the z_i

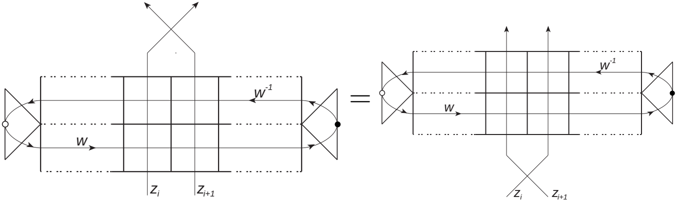
Baxterised 2BTL elements

$\check{R}_i(z) = a(z)$  $+\, b(z)$ 

$\check{K}_0(w,\zeta) = c(qw,\zeta)$  $+\, d(qw,\zeta)$ 

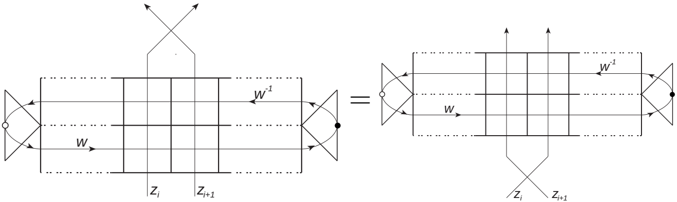
$\check{K}_L(w,\zeta) = c(w,\zeta)$  $+\, d(w,\zeta)$ 

Intertwining relations



$$\check{R}_i(z_i/z_{i+1})T_L = \pi_i(T_L)\check{R}_i(z_i/z_{i+1})$$

Intertwining relations



$$\check{R}_i(z_i/z_{i+1})T_L = \pi_i(T_L)\check{R}_i(z_i/z_{i+1})$$

$$\check{K}_0(1/z_1,\zeta_1)T(w;z_1,\dots) = T(w;1/z_1,\dots)\check{K}_0(1/z_1,\zeta_1)$$
$$\check{K}_L(1/sz_L,\zeta_2)T(w;\dots,z_L) = T(w;\dots,1/s^2z_L)\check{K}_L(1/sz_L,\zeta_2)$$

for $s^4 = 1$.

Part 3:

The q-Knizhnik-Zamolodchikov equation

Derivation of the q-KZ equation

$$\begin{aligned} & \pi_i T_L \pi_i \check{R}_i(z_i/z_{i+1})|\Psi\rangle = \check{R}_i(z_i/z_{i+1}) T_L|\Psi\rangle \\ \Rightarrow \quad & T_L \pi_i \check{R}_i(z_i/z_{i+1})|\Psi\rangle = \pi_i \check{R}_i(z_i/z_{i+1})|\Psi\rangle \end{aligned}$$

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leads to

$$\begin{aligned} \pi_i \check{R}_i(z_i/z_{i+1})|\Psi\rangle &= |\Psi\rangle \\ \check{R}_i(z_i/z_{i+1})|\Psi\rangle &= \pi_i |\Psi\rangle \end{aligned}$$

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The q-KZ equation

$$\begin{aligned} \check{R}_i(z_i/z_{i+1})|\Psi\rangle &= \pi_i|\Psi\rangle \\ \check{K}_0(1/z_1, \zeta_1)|\Psi\rangle &= \pi_0|\Psi\rangle \\ \check{K}_L(1/sz_L, \zeta_2)|\Psi\rangle &= \pi_L|\Psi\rangle \end{aligned}$$

The q-KZ equation

The q-KZ equation

$$e_i|\Psi\rangle = -a_i|\Psi\rangle$$

or

$$\sum_\alpha \psi_\alpha(e_i|\alpha\rangle) = -\sum_\alpha (a_i\psi_\alpha)|\alpha\rangle$$

- Turns into a system of equations for the ψ_α, one for each link pattern α and position i
- 2^L link patterns, $L+1$ positions $\rightarrow 2^L(L+1)$ equations

The q-KZ equation: RHS

Hecke operators

$$a_i \propto (\pi_i - 1)\,[qz_{i+1}/z_i], \qquad {\scriptstyle 1 \le i \le L-1}$$
$$a_0 \propto (\pi_0 - 1)\,k(1/z_1, \zeta_1)$$
$$a_L \propto (\pi_L - 1)\,k(sz_L, 1/s\zeta_2)$$

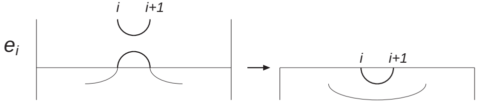
$a_i f = 0 \Rightarrow [qz_{i+1}/z_i]f$ invariant under $z_i \leftrightarrow z_{i+1}$

$a_0 f = 0 \Rightarrow k(1/z_1, \zeta_1)f$ invariant under $z_1 \leftrightarrow 1/z_1$

$a_L f = 0 \Rightarrow k(sz_L, 1/s\zeta_2)f$ invariant under $sz_L \leftrightarrow 1/sz_L$

The q-KZ equation: LHS

e_i acting on a link pattern leads to a small link between i and $i+1$



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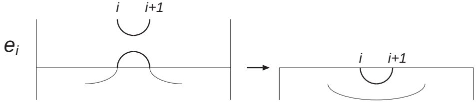

Any α without a small link between i and $i+1$ has

$$a_i \psi_\alpha = 0$$

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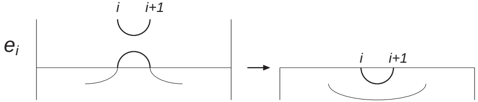

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So ψ_α has a factor of $[qz_i/z_{i+1}]$, symmetric in $z_i \leftrightarrow z_{i+1}$ otherwise.

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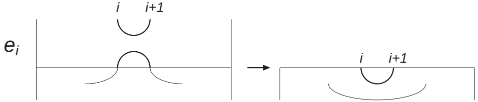

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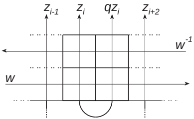
So ψ_α has a factor of $[qz_i/z_{i+1}]$, symmetric in $z_i \leftrightarrow z_{i+1}$ otherwise.
$\Rightarrow \psi_\alpha = 0$ when $z_{i+1} = qz_i$

Part 4:
Recursions and Solutions

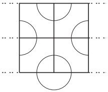
Recursion for Transfer Matrix

$$T_L|_{z_{i+1}=qz_i}\varphi_i|\alpha\rangle =$$

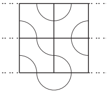


where φ_i inserts a small loop at i

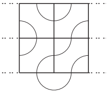
$$a(z_iw)b(qz_iw)\left(a(w/z_i)b(w/qz_i)\right.$$



$$+b(w/z_i)b(w/qz_i)$$



$$\left.+\,a(w/z_i)a(w/qz_i)\right)$$

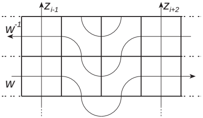


$$= 0$$

Recursion for Transfer Matrix

$$T_L|_{z_{i+1}=qz_i} \circ \varphi_i|\alpha\rangle$$

$$= a(z_i w)b(qz_i w)b(w/z_i)a(w/qz_i)$$



$$= \varphi_i T_{L-2}(\hat{z}_i, \hat{z}_{i+1})|\alpha\rangle$$

where $\hat{z}_j$ is an omission of the argument z_j

Recursion for Eigenstate

$$\begin{aligned} T_L\big|_{z_{i+1}=qz_i} \circ \varphi_i \big|\Psi_{L-2}(\hat{z}_i, \hat{z}_{i+1})\big\rangle &= \varphi_i\, T_{L-2}(\hat{z}_i, \hat{z}_{i+1}) \big|\Psi_{L-2}(\hat{z}_i, \hat{z}_{i+1})\big\rangle \\ &= \varphi_i \big|\Psi_{L-2}(\hat{z}_i, \hat{z}_{i+1})\big\rangle \end{aligned}$$

Recursion for Eigenstate

$$\begin{aligned} T_L\big|_{z_{i+1}=qz_i} \circ \varphi_i |\Psi_{L-2}(\hat{z}_i, \hat{z}_{i+1})\rangle &= \varphi_i \, T_{L-2}(\hat{z}_i, \hat{z}_{i+1}) |\Psi_{L-2}(\hat{z}_i, \hat{z}_{i+1})\rangle \\ &= \varphi_i |\Psi_{L-2}(\hat{z}_i, \hat{z}_{i+1})\rangle \end{aligned}$$

$$|\Psi_L\rangle\big|_{z_{i+1}=qz_i} \propto \varphi_i |\Psi_{L-2}(\hat{z}_i, \hat{z}_{i+1})\rangle$$

with proportionality factor

$$p_i(z_i; z_1, \ldots, \hat{z}_i, \hat{z}_{i+1}, \ldots, z_L)$$

Recursion for components ψ_α

$$\psi^L_{(\cdots(} = \prod_{i=1}^{L} k(z_i, \zeta_1) \prod_{1\leq i<j\leq L} [qz_i/z_j][q/z_iz_j] \times S_1(z_1, \ldots, z_L, 1/z_1, \ldots, 1/z_L)$$

- Can use q-KZ system equations to find $\psi^L_{(\cdots()}$.
- Then use definition for $\psi^{L-2}_{(\cdots(}$ and find p_{L-1}, by
$$\psi^L_{(\cdots()}|_{z_L=qz_{L-1}} = p_{L-1}\psi^{L-2}_{(\cdots(}$$

Recursion for Normalisation

Normalisation

$$Z_L = \langle 0|\Psi\rangle = \sum_\alpha \psi_\alpha$$

$$Z_L|_{z_L = qz_{L-1}} = p_{L-1} Z_{L-2}$$

also, $Z_L = \pi_i Z_L$ for any i, so p_i is the same function for all i. Or,

$$Z_L|_{z_{i+1} = qz_i} = p(z_i; \dots, \hat{z}_i, \hat{z}_{i+1}, \dots) Z_{L-2}(\hat{z}_i, \hat{z}_{i+1})$$

Solution

$$\psi^L_{(\cdots(} = (-1)^{L/2} \prod_{i=1}^{L} k(z_i, \zeta_1) \prod_{1\leq i<j\leq L} k(z_j, z_i)$$
$$\times \chi_\lambda^{(L+1)}(z_1^2, \ldots, z_L^2, (s\zeta_2)^2)$$

$$\psi^L_{)\cdots)} = (-1)^{L/2} \prod_{i=1}^{L} k(1/sz_i, 1/s^2\zeta_2) \prod_{1\leq i<j\leq L} k(1/sz_i, 1/sz_j)$$
$$\times \chi_\lambda^{(L+1)}((s\zeta_1)^2, (sz_1)^2, \ldots, (sz_L)^2),$$

$s^4 = 1.$

Solution

$$\chi_\lambda^{(L)}(z_1, \ldots, z_L) = \frac{\left| z_i^{\lambda_j + L - j + 1} - z_i^{-\lambda_j - L + j - 1} \right|_{1 \leq i,j \leq L}}{\left| z_i^{L - j + 1} - z_i^{-L + j - 1} \right|_{1 \leq i,j \leq L}}$$

of degree λ^L in its arguments,

$$\lambda^L = (\ldots, 2, 2, 1, 1, 0, 0);$$

and the ψ are of degree $2\lambda^{L+1} + \lambda^{L+2}$ in $(z_1^2, \ldots, z_L^2)$.

Solution

Z_L has

- the same recursion
- the same degree $(2\lambda^{(L+1)} + \lambda^{(L+2)})$
- the same values for $L = 2$ and $L = 3$

as

$$\chi_\lambda^{L+1}(\zeta_1^2, z_1^2, \dots, z_L^2) \\ \times \chi_\lambda^{L+1}(z_1^2, \dots, z_L^2, \zeta_2^2) \\ \times \chi_\lambda^{L+2}(\zeta_1^2, z_1^2, \dots, z_L^2, \zeta_2^2),$$

Closing Remarks

- Exact expression for two components of vector for general L. Can use qKZ to find some others
- Exact expression for normalisation of vector - possible connection to FPL models when $z \to 1$.
- de Gier, J., Shigechi, K. and Ponsaing, A., *Exact finite size groundstate of the* $O(n = 1)$ *loop model with open boundaries*, arXiv:0901.2961