The Brauer loop scheme with boundaries

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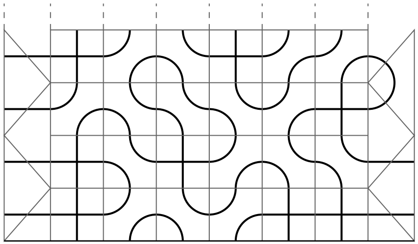
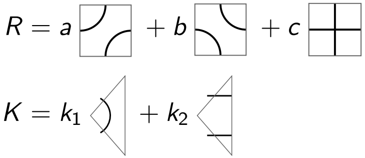
Brauer loop model

The Brauer loop model

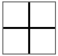
Loop model on a semi-infinite lattice. Closed loops are ignored ($\tau = 1$), focus on connectivities.

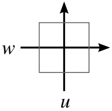




R Matrix

Introduce inhomogeneities.

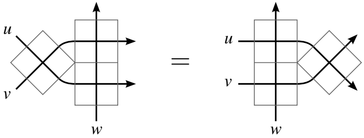
$R(w-u) = a(w-u)$  $+\, b(w-u)$  $+\, c(w-u)$ 

$=$ 

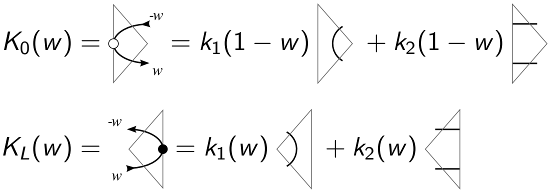
Probability of configurations on a face:

$$a(z) = \frac{2(1-z)}{(1+z)(2-z)}, \quad b(z) = \frac{2z}{(1+z)(2-z)}, \quad c(z) = \frac{z(1-z)}{(1+z)(2-z)}$$

Chosen so that Yang–Baxter equation holds:



K Matrices



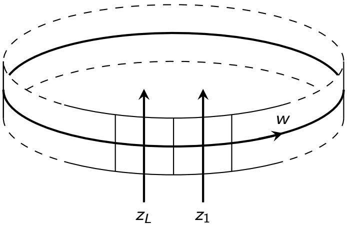
Inhomogeneous probabilities (depending on boundary conditions):

$$k_1(w) = \frac{1-2w}{1+2w}, \quad k_2(w) = \frac{4w}{1+2w}$$

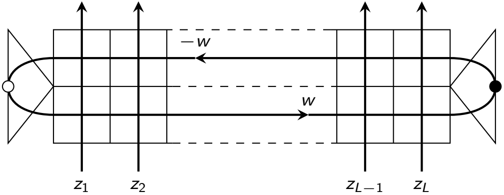
Chosen so that boundary YBE holds.

Transfer Matrix

Probabilities of configurations on one row of the lattice (periodic):

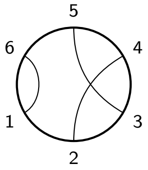
$T(w; z_1, \ldots, z_L) =$ 

or two rows (all other BCs):

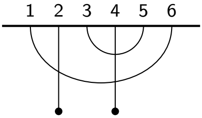
$T(w; z_1, \ldots, z_L) =$ 

LP_L^a is the set of all link patterns with boundary condition a:

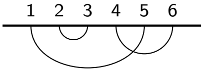
Periodic



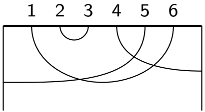
Identified



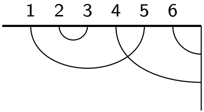
Closed



Open



Mixed



Ground state eigenvector

Steady state of stochastic process is ground state of transfer matrix

$$|\Psi(z_1, \ldots, z_L)\rangle = \sum_{\alpha \in \mathsf{LP}_L} \psi_\alpha(z_1, \ldots, z_L)|\alpha\rangle$$

Unique, with eigenvalue of 1:

$$T(w; z_1, \ldots, z_L)\ |\Psi(z_1, \ldots, z_L)\rangle = |\Psi(z_1, \ldots, z_L)\rangle$$

Sum rule (eigenvector normalization):

$$Z_L = \sum_\alpha \psi_\alpha$$

Calculating the eigenvector

Two properties define $|\Psi\rangle$ uniquely:

- **Quantum Knizhnik–Zamolodchikov (qKZ) equation:**

$$\begin{aligned}\check{R}_i(z_i - z_{i+1})|\Psi(\dots, z_i, z_{i+1}, \dots)\rangle &= |\Psi(\dots, z_{i+1}, z_i, \dots)\rangle \\ \check{K}_0(-z_1)|\Psi(z_1, \dots)\rangle &= |\Psi(-z_1, \dots)\rangle \\ \check{K}_L(z_L)|\Psi(\dots, z_L)\rangle &= |\Psi(\dots, -z_L)\rangle\end{aligned}$$

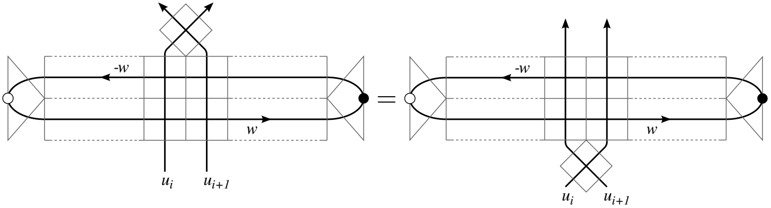
where $\check{R}_i$ is tilted version of R.

- **Recursion:**

$$\begin{aligned}|\Psi_L(\dots, z_i, z_i + 1, \dots)\rangle &= p(z_i; \dots, \hat{z}_i, \hat{z}_{i+1}, \dots)\varphi_i|\Psi_{L-2}(\dots, \hat{z}_i, \hat{z}_{i+1}, \dots)\rangle \\ |\Psi_L\left(1/2, z_2, \dots, z_L\right)\rangle &= p_0(z_2, \dots, z_L)\varphi_0|\Psi_{L-1}(z_2, \dots, z_L)\rangle \\ |\Psi_L\left(z_1, \dots, z_{L-1}, -1/2\right)\rangle &= p_L(z_1, \dots, z_{L-1})\varphi_L|\Psi_{L-1}(z_1, \dots, z_{L-1})\rangle\end{aligned}$$

qKZ equation (bulk)

YBE implies interlacing relation:



Equivalently,

$$\check{R}_i(z_i - z_{i+1})T(w,\dots,z_i,z_{i+1},\dots) = T(w,\dots,z_{i+1},z_i,\dots)\check{R}_i(z_i - z_{i+1})$$

Leading to bulk qKZ equation:

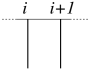
$$\check{R}_i(z_i - z_{i+1})|\Psi(\dots,z_i,z_{i+1},\dots)\rangle = |\Psi(\dots,z_{i+1},z_i,\dots)\rangle$$

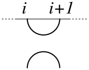
qKZ equation (bulk)

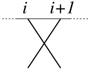
$$\check{R}_i(z_i - z_{i+1})|\Psi(\ldots, z_i, z_{i+1}, \ldots)\rangle = |\Psi(\ldots, z_{i+1}, z_i, \ldots)\rangle$$

$\check{R}_i$ in terms of Brauer algebra generators:

$$\check{R}_i(z_i - z_{i+1}) = a_i + b_i\ e_i + c_i\ f_i$$

$= a(z_i - z_{i+1})$ 

$+ b(z_i - z_{i+1})$ 

$+ c(z_i - z_{i+1})$ 

$$\sum_\alpha (a_i + b_i e_i + c_i f_i)\psi_\alpha|\alpha\rangle = \sum_\alpha \pi_i \psi_\alpha |\alpha\rangle \qquad \big(= |\Psi(z_{i+1}, z_i)\rangle \big)$$

If α has no link from i to $i+1$, then ψ_α has a factor of $(1 + z_i - z_{i+1})$.

qKZ equation (bulk)

Three cases for $|\alpha\rangle$:

1. $f_i|\alpha\rangle = |\alpha\rangle$ & α has no loop from i to $i+1$,

$$(a_i + c_i)\psi_\alpha = \pi_i \psi_\alpha$$

qKZ equation (bulk)

Three cases for $|\alpha\rangle$:

1. $f_i|\alpha\rangle = |\alpha\rangle \quad \& \quad \alpha$ has no loop from i to $i+1$,

$$(a_i + c_i)\psi_\alpha = \pi_i \psi_\alpha$$

More detail:

$$(a_i + c_i)\psi_\alpha(z_i, z_{i+1}) = \psi_\alpha(z_{i+1}, z_i)$$

$$\frac{(z_{i+1} - z_i + 1)(z_{i+1} - z_i - 2)}{(z_i - z_{i+1} + 1)(z_i - z_{i+1} - 2)}\psi_\alpha(z_i, z_{i+1}) =$$

$$\Rightarrow \quad \psi_\alpha(z_i, z_{i+1}) = (z_i - z_{i+1} + 1)(z_i - z_{i+1} - 2)S^{\{i,i+1\}}(z_i, z_{i+1})$$

qKZ equation (bulk)

Three cases for $|\alpha\rangle$:

2. $f_i|\alpha\rangle = |\beta\rangle \neq |\alpha\rangle, \quad \Rightarrow \quad \alpha$ has no loop from i to $i+1$,

$$a_i\psi_\alpha + c_i\psi_\beta = \pi_i\psi_\alpha$$
$$a_i\psi_\beta + c_i\psi_\alpha = \pi_i\psi_\beta$$

qKZ equation (bulk)

Three cases for $|\alpha\rangle$:

2. $f_i|\alpha\rangle = |\beta\rangle \neq |\alpha\rangle, \quad \Rightarrow \quad \alpha$ has no loop from i to $i+1$,

$$a_i\psi_\alpha + c_i\psi_\beta = \pi_i\psi_\alpha$$
$$a_i\psi_\beta + c_i\psi_\alpha = \pi_i\psi_\beta$$

More detail:

$$(1 - z_i + z_{i+1})\big(2\psi_\alpha(z_i, z_{i+1}) + (z_i - z_{i+1})\psi_\beta(z_i, z_{i+1})\big)$$
$$= (1 + z_i - z_{i+1})(2 - z_i + z_{i+1})\psi_\alpha(z_{i+1}, z_i)$$

Taken at $z_i = z_{i+1} + 1$, gives $\psi_\alpha(z_{i+1}, z_{i+1} + 1) = 0$, so

$$\psi_\alpha = (1 + z_i - z_{i+1})f_\alpha.$$

qKZ equation (bulk)

Three cases for $|\alpha\rangle$:

3. α has a loop from i to $i+1$, $\quad \Rightarrow \quad f_i|\alpha\rangle = |\alpha\rangle$,

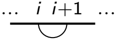
$$(a_i + c_i)\psi_\alpha + b_i \sum_{\gamma : e_i|\gamma\rangle = |\alpha\rangle} \psi_\gamma = \pi_i \psi_\alpha$$

qKZ equation (bulk)

Three cases for $|\alpha\rangle$:

3. α has a loop from i to $i+1$, $\quad\Rightarrow\quad f_i|\alpha\rangle = |\alpha\rangle$,

$$(a_i + c_i)\psi_\alpha + b_i \sum_{\gamma: e_i|\gamma\rangle = |\alpha\rangle} \psi_\gamma = \pi_i \psi_\alpha$$

Consequence: $\psi_\alpha(z_{i+1} = z_i + 1) = 0$ unless $\alpha =$ 

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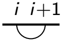
So

$$|\Psi_L(z_{i+1} = z_i + 1)\rangle = \varphi_i |\Phi_{L-2}\rangle,$$

where $\varphi_i =$ 

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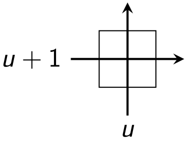
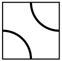
Recursion (bulk)

Note

$R(1) =$  $= $ 

$\check{R}_i(1) = e_i$

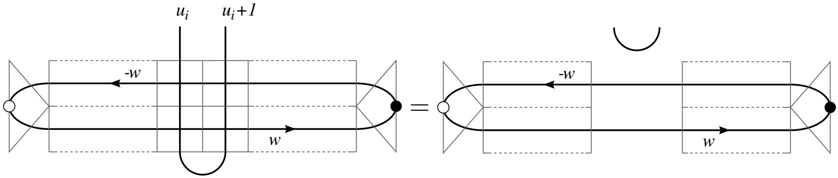
Interlacing condition (rewritten using unitarity of R-matrix)

$$T(w, \ldots, z_i, z_{i+1}, \ldots)\check{R}_i(z_{i+1} - z_i) = \check{R}_i(z_{i+1} - z_i)T(w, \ldots, z_{i+1}, z_i, \ldots)$$

Taken at $z_{i+1} = z_i + 1$,

$$T(w, \ldots, z_i, z_i + 1, \ldots)e_i = e_i T(w, \ldots, z_i + 1, z_i, \ldots)$$

Recursion (bulk)



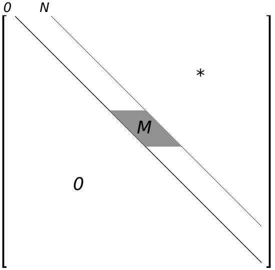
$$
\begin{aligned}
|\Psi_L(z_{i+1} = z_i + 1)\rangle &= \varphi_i |\Phi_{L-2}\rangle \qquad \text{(from qKZ)} \\
&= T_L(w, \dots, z_i, z_i + 1, \dots) \varphi_i |\Phi_{L-2}\rangle \\
&= \varphi_i T_{L-2}(w, \dots, \hat{z}_i, \hat{z}_{i+1}, \dots) |\Phi_{L-2}\rangle
\end{aligned}
$$

$$
\Rightarrow \quad |\Psi_{L-2}(\hat{z}_i, \hat{z}_{i+1})\rangle \propto |\Phi_{L-2}\rangle.
$$

Brauer loop scheme

The Brauer loop scheme (periodic)

$\mathcal{M}_N^{\mathrm{p}} =$ Infinite upper-triangular (N, N)-periodic strip matrices:



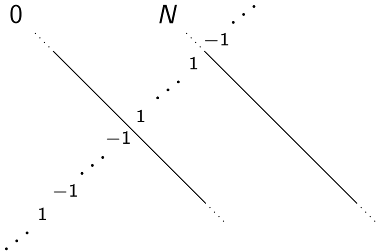
$$E_N^{\mathrm{p}} = \left\{ M \in \mathcal{M}_N^{\mathrm{p}} \;\middle|\; (M^2)_{ij} = 0 \text{ for } i \le j < i + N \right\}$$

The Brauer loop scheme (identified boundaries)

$N = 2L$.

Symplectic form:

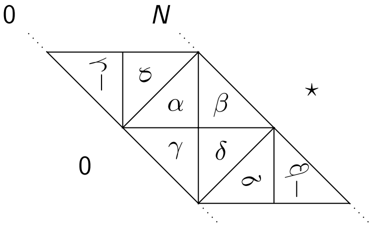
$J =$



The Brauer loop scheme (identified boundaries)

$$E_N^{\mathsf{i}} = E_N^{\mathsf{p}} \cap \left\{ M \in \mathcal{M}_N^{\mathsf{p}} \;\middle|\; M = J M^T J \right\}$$

$E_N^{\mathsf{i}} \ni M =$



The Brauer loop scheme

Claims:

1. The irreducible components (varieties) of E_N^a are labelled in a prescribed way by link patterns, E_α^a.
2. The multidegrees of the varieties (defined later) satisfy the qKZ equation.
3. The multidegrees satisfy the same recursions as the components of the Brauer eigenvector.

Therefore:

- The multidegree of E_α^a is proportional to the component of the eigenvector ψ_α.

Proofs: Knutson & Zinn-Justin 2007 (periodic), P. & Zinn-Justin forthcoming (identified & rest)

Claim 1

Prescription

Irreducible component E_α^a = unique highest dimensional component of

$$P_\alpha = \left\{ M \in E_N^a \;\middle|\; \left(M^2\right)_{i,i+N} = \left(M^2\right)_{\alpha(i),\alpha(i)+N} \right\}.$$

Claim 1: Example $N = 3$, periodic

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} & * & * & * \\ 0 & m_{22} & m_{23} & m_{21} & * & * \\ 0 & 0 & m_{33} & m_{31} & m_{32} & * \end{pmatrix}$$

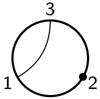
$E_N^{\mathrm{p}} : (M^2 = 0)$

$$\begin{cases} m_{11}^2 = 0, & m_{11}m_{12} + m_{12}m_{22} = 0, & m_{11}m_{13} + m_{12}m_{23} + m_{13}m_{33} = 0 \\ m_{22}^2 = 0, & m_{22}m_{23} + m_{23}m_{33} = 0, & m_{22}m_{21} + m_{23}m_{31} + m_{21}m_{11} = 0 \\ m_{33}^2 = 0, & m_{33}m_{31} + m_{31}m_{11} = 0, & m_{33}m_{32} + m_{31}m_{12} + m_{32}m_{22} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} m_{11} = 0, & m_{12}m_{23} = 0 \\ m_{22} = 0, & m_{23}m_{31} = 0 \\ m_{33} = 0, & m_{31}m_{12} = 0 \end{cases}$$

Claim 1: Example $N = 3$, periodic

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} & * & * & * \\ 0 & m_{22} & m_{23} & m_{21} & * & * \\ 0 & 0 & m_{33} & m_{31} & m_{32} & * \end{pmatrix}$$

Take $\alpha = (13)(2) =$  .

$$P_\alpha = \left\{ M \in E_N^{\mathrm{p}} \;\middle|\; \left(M^2\right)_{1,4} = \left(M^2\right)_{3,6} \right\}$$

$$m_{12}m_{21} + m_{13}m_{31} = m_{31}m_{13} + m_{32}m_{23}$$
$$m_{12}m_{21} = m_{32}m_{23}$$

Claim 1: Example $N = 3$, periodic

$$P_\alpha = \mathcal{M}_N^{\mathsf{p}} / \langle m_{11},\, m_{22},\, m_{33},\, m_{12}m_{23},\, m_{23}m_{31},\, m_{31}m_{12},\, m_{12}m_{21} - m_{32}m_{23} \rangle$$

Decomposes into:

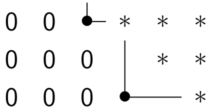
$$\begin{array}{l}
\mathcal{M}_N^{\mathsf{p}} / \langle m_{11},\, m_{22},\, m_{33},\, m_{12},\, m_{23} \rangle \text{, dim 4} \\
\mathcal{M}_N^{\mathsf{p}} / \langle m_{11},\, m_{22},\, m_{33},\, m_{12},\, m_{31},\, m_{32} \rangle \text{, dim 3} \\
\mathcal{M}_N^{\mathsf{p}} / \langle m_{11},\, m_{22},\, m_{33},\, m_{12},\, m_{31},\, m_{23} \rangle \text{, dim 3} \\
\mathcal{M}_N^{\mathsf{p}} / \langle m_{11},\, m_{22},\, m_{33},\, m_{23},\, m_{31},\, m_{21} \rangle \text{, dim 3}
\end{array}$$

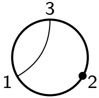
Thus,

$$E_\alpha^{\mathsf{p}} \ni M = \begin{pmatrix} 0 & 0 & m_{13} & * & * & * \\ 0 & 0 & 0 & m_{21} & * & * \\ 0 & 0 & 0 & m_{31} & m_{32} & * \end{pmatrix}$$

Claim 1: Example $N = 3$, periodic

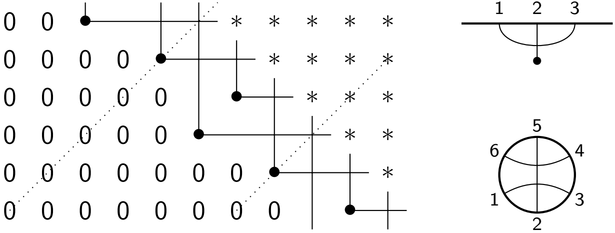
$$\begin{matrix} 0 & 0 & m_{13} & * & * & * \\ 0 & 0 & 0 & m_{21} & * & * \\ 0 & 0 & 0 & m_{31} & m_{32} & * \end{matrix} \quad \rightarrow$$





Claim 1: Example $N = 6$, identified

$$
\begin{array}{ccccccccccc}
0 & 0 & m_{13} & m_{14} & m_{15} & m_{16} & * & * & * & * & * \\
0 & 0 & 0 & 0 & m_{25} & m_{15} & m_{21} & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & m_{14} & m_{31} & m_{32} & * & * & * \\
0 & 0 & 0 & 0 & 0 & -m_{13} & m_{41} & m_{42} & m_{43} & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & m_{52} & m_{42} & -m_{32} & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m_{41} & -m_{31} & -m_{21}
\end{array}
$$



Multidegrees defined

To any variety X belonging to a space W, associate a polynomial mdeg(X) in a canonical way. Let H be a hyperplane in W.

- If $X = W$, $\text{mdeg}_W(X)$=1.
- If X_i are top-dimensional components of X of multiplicity m_i, then $\text{mdeg}_W(X) = \sum_i m_i \ \text{mdeg}_W(X_i)$.
- If X irreducible and $X \not\subset H$, then $\text{mdeg}_W(X)$ =$\text{mdeg}_H(X \cap H)$.
- If X irreducible and $X \subset H$, then $\text{mdeg}_W(X)$ =$\text{mdeg}_H(X)\cdot$(weight of torus action on W/H) .

Multidegrees defined

For a chain of hyperplanes so that $X = H_0 \subset \cdots \subset H_N = W$,

$$\mathrm{mdeg}_W(X) = \prod_{i=1}^{N} \mathrm{wt}_T(H_i/H_{i-1}).$$

- Torus action: $M_{ij} \mapsto \lambda x_i x_j^{-1} M_{ij} = \exp(A + z_i - z_j) M_{ij}$
- Weight of action: $A + z_i - z_j$.

Example

$\mathrm{mdeg}_{\mathcal{M}_N^{\mathrm{p}}}(\mathcal{M}_N^{\mathrm{p}} / \langle m_{12}, m_{13} \rangle) = (A + z_1 - z_2)(A + z_1 - z_3)$.

The Brauer loop scheme

Claim 2 Multidegrees satisfy the qKZ equation (Hard)

Claim 3 Multidegrees satisfy the recursion at $z_{i+1} = z_i + 1$ (Not so hard)

Claim 3

$$\left[\begin{array}{cc|cc|cc|c} m_{11} & m_{12} & m_{13} & m_{14} & * & * & * \\ 0 & m_{22} & m_{23} & m_{24} & m_{21} & * & * \\ \hline 0 & 0 & m_{33} & m_{34} & m_{31} & m_{32} & * \\ 0 & 0 & 0 & m_{44} & m_{41} & m_{42} & m_{43} \\ \hline 0 & 0 & 0 & 0 & m_{11} & m_{12} & m_{13} \end{array}\right]$$

$E_N^{\mathrm{p}} : (M^2 = 0)$

$$0 = m_{33}^2$$
$$0 = m_{33} m_{34} + m_{34} m_{44}$$
$$0 = m_{33} m_{31} + m_{34} m_{41} + m_{31} m_{11}$$
$$0 = m_{33} m_{32} + m_{34} m_{42} + m_{31} m_{12} + m_{32} m_{22}$$
$$0 = m_{11} m_{14} + m_{12} m_{24} + m_{13} m_{34} + m_{14} m_{44}$$
$$0 = m_{22} m_{24} + m_{23} m_{34} + m_{24} m_{44}$$

Claim 3

$$\left[\begin{array}{cc|cc|cc|c} m_{11} & m_{12} & m_{13} & m_{14} & * & * & * \\ 0 & m_{22} & m_{23} & m_{24} & m_{21} & * & * \\ \hline 0 & 0 & m_{33} & \color{red}{m_{34}} & m_{31} & m_{32} & * \\ 0 & 0 & 0 & m_{44} & m_{41} & m_{42} & m_{43} \\ \hline 0 & 0 & 0 & 0 & m_{11} & m_{12} & m_{13} \end{array}\right]$$

$m_{34} \to \infty$ (If not already $= 0$)

$$\begin{aligned} 0 &= m_{33}^2 \\ 0 &= m_{34}(m_{33} + m_{44}) \\ 0 &= m_{34} m_{41} \\ 0 &= m_{34} m_{42} \\ 0 &= m_{13} m_{34} \\ 0 &= m_{23} m_{34} \end{aligned}$$

Claim 3

$$\left[\begin{array}{cc|cc|cc|c} 0 & m_{12} & 0 & m_{14} & * & * & * \\ 0 & 0 & 0 & m_{24} & m_{21} & * & * \\ \hline 0 & 0 & 0 & \infty & m_{31} & m_{32} & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & m_{12} & 0 \end{array}\right]$$

$m_{34} \to \infty$ (If not already $= 0$)

$$0 = m_{33}$$
$$0 = m_{44}$$
$$0 = m_{41}$$
$$0 = m_{42}$$
$$0 = m_{13}$$
$$0 = m_{23}$$

Claim 3

$$\left[\begin{array}{cc|cc|cc|c} 0 & m_{12} & 0 & * & * & * & * \\ 0 & 0 & 0 & * & m_{21} & * & * \\ \hline 0 & 0 & 0 & \infty & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & m_{12} & 0 \end{array}\right]$$

$m_{34} \to \infty$ (If not already $= 0$)

$$0 = m_{33}$$
$$0 = m_{44}$$
$$0 = m_{41}$$
$$0 = m_{42}$$
$$0 = m_{13}$$
$$0 = m_{23}$$

Claim 3

$$\left[\begin{array}{cc|cc} 0 & m_{12} & * & * \\ 0 & 0 & m_{21} & * \\ \hline 0 & 0 & 0 & m_{12} \end{array}\right]$$

$$\mathrm{mdeg}(X_N)|_{m_{i,i+1}\to\infty} = \prod_{k\neq i,i+1} (A + z_k - z_i)(A + z_{i+1} - z_k)\,\mathrm{mdeg}(X_{N-2}).$$

Taking $m_{i,i+1} \to \infty$ corresponds to taking $z_{i+1} = z_i + A$. By rescaling, A can be set to 1 and the recursion follows.

Conclusion

Integrable Model	$\leftrightarrow$	**Algebraic Variety**
Brauer loop model with BCs	$\leftrightarrow$	Brauer loop scheme with symmetries
Link patterns	$\leftrightarrow$	Irreducible components
Eigenvector components	$\leftrightarrow$	Multidegrees of irreducible components
Eigenvector normalization	$\leftrightarrow$	Multidegree of scheme