A Solution of the q-deformed Knizhnik-Zamolodchikov Equation

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December 2, 2008

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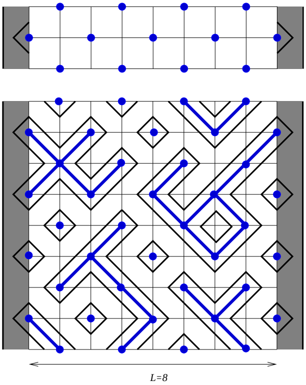
Part 1:

Bond percolation and the 2 boundary Temperley-Lieb algebra (2BTL)

Bond percolation model

Model of bond distributions on a 2−D grid



Can ignore:

- Closed loops
- Links attaching a boundary to itself
- Loops attached to both boundaries

2BTL - A Link Pattern Representation

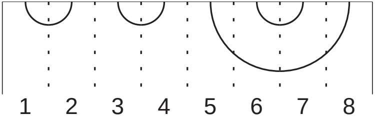


Figure: $|\alpha\rangle$, link pattern of size 8

Shorthand notation: ()()(()).

There are 2^L link patterns in a system of size L.

Temperley-Lieb Generators

$$\{e_i;\ i = 0, \dots, L\}$$

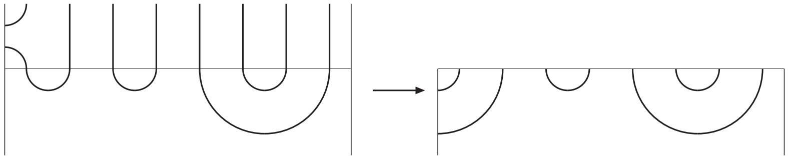


Figure: $e_0|\alpha\rangle$

e_0 acts between the left boundary and position 1

Temperley-Lieb Generators

$$\{e_i;\ i = 0, \ldots, L\}$$

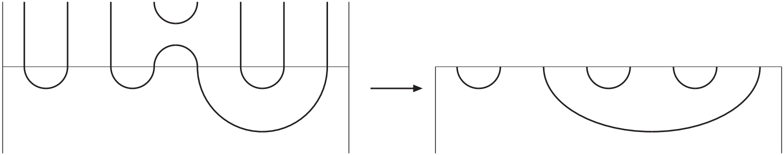


Figure: $e_4|\alpha\rangle$

e_i acts between position i and position $i+1$

Temperley-Lieb Generators

$$\{e_i;\ i = 0, \dots, L\}$$

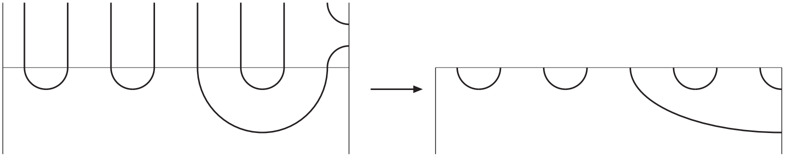


Figure: $e_8|\alpha\rangle$

e_L acts between position L and the right boundary

Part 2:

The q-deformed Knizhnik-Zamolodchikov equation

Eigenstate vector of transfer matrix

Definition

$$|\Psi\rangle = \sum_{\alpha} \psi_{\alpha}(z_1, \ldots, z_L)|\alpha\rangle$$

- Sum runs over link patterns α of size L
- ψ_{α} are (polynomial) functions of L variables

The qKZ equation

The qKZ equation

$$e_i|\Psi\rangle = -a_i|\Psi\rangle$$

or

$$\sum_\alpha \psi_\alpha(e_i|\alpha\rangle) = -\sum_\alpha (a_i\psi_\alpha)|\alpha\rangle$$

- Turns into a system of equations for the ψ_α, one for each link pattern α and position i
- 2^L link patterns, $L+1$ positions $\rightarrow 2^L(L+1)$ equations

The qKZ equation: RHS

Hecke operators

$$a_i \propto (\pi_i - 1)\,[qz_{i+1}/z_i], \qquad {\scriptstyle 1\le i\le L-1}$$
$$a_0 \propto (\pi_0 - 1)\, k(1/z_1, \zeta_1)$$
$$a_L \propto (\pi_L - 1)\, k(sz_L, 1/s\zeta_2)$$

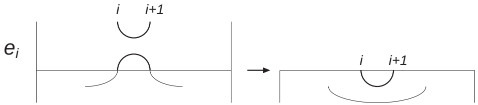
$a_i f = 0 \Rightarrow [qz_{i+1}/z_i]f$ invariant under $z_i \leftrightarrow z_{i+1}$

$a_0 f = 0 \Rightarrow k(1/z_1, \zeta_1)f$ invariant under $z_1 \leftrightarrow 1/z_1$

$a_L f = 0 \Rightarrow k(sz_L, 1/s\zeta_2)f$ invariant under $sz_L \leftrightarrow 1/sz_L$

The qKZ equation: LHS

e_i acting on a link pattern leads to a small link between i and $i+1$



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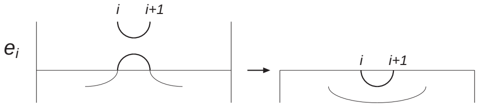

Any α without a small link between i and $i+1$ has

$$a_i \psi_\alpha = 0$$

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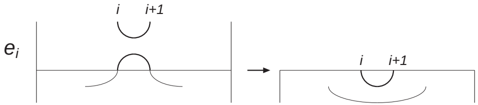

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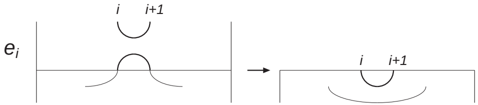

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So $[qz_{i+1}/z_i]\psi_\alpha$ must be symmetric in $z_i \leftrightarrow z_{i+1}$
$\Rightarrow \psi_\alpha$ has a factor of $[qz_i/z_{i+1}]$, symmetric otherwise.

Part 3:
Solutions

Properties of solution: Symmetries

Conditions on ψ_α:

i. α has no link from i to $i+1$
$\rightarrow \psi_\alpha = [qz_i/z_{i+1}]S_\alpha(z_i, z_{i+1})$,

ii. α has no link from the left boundary to 1
$\rightarrow \psi_\alpha = k(z_1, \zeta_1)S_\alpha(z_1, 1/z_1)$,

iii. α has no link from L to the right boundary
$\rightarrow \psi_\alpha = k(1/sz_L, 1/s\zeta_2)S_\alpha(sz_L, 1/sz_L)$.

Condition (i) means that most components disappear when $z_{i+1} = qz_i$.

Solution for general L

$$\psi^L_{(\cdots(} = \prod_{i=1}^{L} k(z_i, \zeta_1) \prod_{1\le i<j\le L} [qz_i/z_j][q/z_i z_j] \times S_1(z_1, \ldots, z_L, 1/z_1, \ldots, 1/z_L)$$

$$\psi^L_{)\cdots)} = \prod_{i=1}^{L} k(1/sz_i, 1/s\zeta_2) \prod_{1\le i<j\le L} [qz_i/z_j][qs^2 z_i z_j] \times S_2(sz_1, \ldots, sz_L, 1/sz_1, \ldots, 1/sz_L)$$

Solution for general L

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Symplectic character

For finite L, have found a special form for S_1 and S_2;

$$S_1 = (-1)^{L/2}\chi_\lambda^{(L+1)}(z_1^2, \dots, z_L^2, \zeta_2^2)$$
$$S_2 = (-1)^{L/2}\chi_\lambda^{(L+1)}((s\zeta_2)^2, (sz_1)^2, \dots, (sz_L)^2)$$

$\chi_\lambda^{(L)}$ is of degree λ^L in its arguments,

$$\lambda^L = (\dots, 3, 2, 2, 1, 1, 0, 0)$$

Recursion:

$$\chi_\lambda^{(L)}(z_1^2, \dots, z_L^2)|_{z_{i+1}=qz_i} \propto \chi_\lambda^{(L-2)}(z_1^2, \dots, \hat{z}_i^2, \hat{z}_{i+1}^2, \dots, z_L^2)$$

Recursion

It is easy to show that

$$\psi^{L}_{(\cdots()}|_{z_L=qz_{L-1}} = p_{L-1}(z_{L-1}; z_1, \ldots, z_{L-2})\psi^{L-2}_{(\cdots(}$$

and it can also be shown that

$$|\Psi^L\rangle|_{z_{i+1}=qz_i} \propto \varphi_i|\Psi^{L-2}\rangle$$
$$\Rightarrow \psi^{L}_{\cdots()}|_{z_L=q_{L-1}} = p_{L-1}\psi^{L-2}_{\cdots}$$

Recursion

Definition

$$Z_L = \sum_{\alpha} \psi_\alpha$$

$$\pi_i Z_L = Z_L, \qquad 0 \leq i \leq L$$

So we have

$$Z_L(z_1, \ldots, z_i, qz_i, \ldots, z_L) = p_i Z_{L-2}(z_1, \ldots, z_{i-1}, z_{i+2}, \ldots, z_L)$$

and

$$p_i = (-1)^L k(z_i, \zeta_1)^2 k(z_i, \zeta_2)^2 \prod_{j \neq i, i+1}^{L-2} \left(k(z_i, z_j)^3\right)$$

Solution for Z_L

The product of symplectic characters

$$X_L = \chi_\lambda^{L+1}(\zeta_1^2, z_1^2, \dots, z_L^2) \\ \times \chi_\lambda^{L+1}(z_1^2, \dots, z_L^2, \zeta_2^2) \\ \times \chi_\lambda^{L+2}(\zeta_1^2, z_1^2, \dots, z_L^2, \zeta_2^2)$$

has a recursion

$$X_L|_{z_{i+1}=qz_i} = p_i X_{L-2}$$

Solution for Z_L

Z_L and X_L have

- The same recursion
- The same degree $(2\lambda^{(L+1)} + \lambda^{(L+2)})$
- The same values for $L = 2$ and $L = 3$

So

$$Z_L = X_L$$

Part 4:
Closing Remarks

Conclusions and related problems

- Found an exact expression for the groundstate of a finite system
- Equivalent to the Hamiltonian groundstate when $z_i \to 1$
- $\mathrm{O}(n = 1)$ loop model
- Numbers of fully packed loop configurations with different boundary conditions

Example: $L = 2$

Set $|1\rangle = |((\rangle$, $|2\rangle = |()\rangle$, $|3\rangle = |)(\rangle$, and $|4\rangle = |))\rangle$.

For $L = 2$, $i = 0$:

$$-\sum_{\alpha} a_0 \psi_\alpha |\alpha\rangle = \psi_1 e_0 |1\rangle + \psi_2 e_0 |2\rangle + \psi_3 e_0 |3\rangle + \psi_4 e_0 |4\rangle$$
$$= (\psi_1 + \psi_3)|3\rangle + (\psi_2 + \psi_4)|4\rangle$$

leads to

$$a_0\psi_1 = a_0\psi_2 = 0$$
$$-a_0\psi_3 = \psi_1 + \psi_3$$
$$-a_0\psi_4 = \psi_2 + \psi_4$$

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$$= (\psi_1 + \psi_3)|3\rangle + (\psi_2 + \psi_4)|4\rangle$$

or, with $s_i = -1 - a_i$,

$$a_0 \psi_1 = a_0 \psi_2 = 0$$
$$s_0 \psi_3 = \psi_1$$
$$s_0 \psi_4 = \psi_2$$

Example: $L = 2$

Considering $i = 1$ and 2 in turn gives the 12 system equations:

$$\begin{aligned} 0 &= a_0\psi_1 = a_0\psi_2 \\ &= a_2\psi_2 = a_2\psi_4 \\ &= a_1\psi_1 = a_1\psi_3 = a_1\psi_4 \end{aligned}$$

$$\begin{aligned} s_0\psi_3 &= \psi_1 \qquad & s_2\psi_1 &= \psi_2 \\ s_0\psi_4 &= \psi_2 \qquad & s_2\psi_3 &= \psi_4 \end{aligned}$$

$$s_1\psi_2 = \psi_1 + \psi_3 + \psi_4.$$