

Separation of Variables for the Symplectic Character

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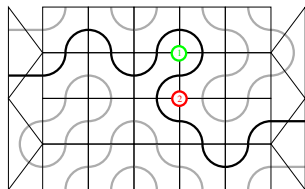
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Work done in collaboration with Jan de Gier
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Lattice Model - $O(n=1)$ loop model

see talk by Jan de Gier, Wednesday



Correlation function:

$$X_L^{(k)} = c_L z_k \frac{\partial}{\partial z_k} \log \left[\frac{\chi_\lambda(\zeta_1^2, z_1^2, \dots, z_L^2) \chi_\lambda(z_1^2, \dots, z_L^2, \zeta_2^2)}{\chi_\lambda(z_1^2, \dots, z_L^2) \chi_\lambda(\zeta_1^2, z_1^2, \dots, z_L^2, \zeta_2^2)} \right]$$

Partition function:

$$Z_L = \chi_\lambda(z_1^2, \dots, z_L^2) \chi_\lambda(\zeta_1^2, z_1^2, \dots, z_L^2) \\ \times \chi_\lambda(z_1^2, \dots, z_L^2, \zeta_2^2) \chi_\lambda(\zeta_1^2, z_1^2, \dots, z_L^2, \zeta_2^2)$$

Q-operator method (Kuznetsov, Mangazeev, Sklyanin 2003)

Polynomials P_λ are eigenfunctions of set of commuting Hamiltonians H_j ,

$$H_j P_\lambda(x_1, \dots, x_L) = h_j(\lambda) P_\lambda(x_1, \dots, x_L)$$

Want a version where the polynomial is factorised, ie

$$\tilde{H}_j \prod_{i=1}^L q_\lambda(x_i) = h_j \prod_{i=1}^L q_\lambda(x_i)$$

Want to find a 'Separating operator' \mathcal{S} ,

$$\mathcal{S} P_\lambda(x_1, \dots, x_L) = \prod_{i=1}^L q_\lambda(x_i) P_\lambda(1, \dots, 1),$$

which allows transformation of spectral problem;

$$(\mathcal{S} H_j \mathcal{S}^{-1}) \mathcal{S} P_\lambda(x_1, \dots, x_L) = h_j(\lambda) \mathcal{S} P_\lambda(x_1, \dots, x_L)$$

Q-operator method

$$SP_\lambda(x_1, \dots, x_L) = \prod_{i=1}^L q_\lambda(x_i) P_\lambda(1, \dots, 1)$$

Introduce Q-operator:

$$Q_z P_\lambda(x_1, \dots, x_L) = q_\lambda(z) P_\lambda(x_1, \dots, x_L)$$

Separating operator is a chain of Q-operators

$$S = (Q_{z_1} \dots Q_{z_L}) \Big|_{x_1, \dots, x_L=1} \Big|_{z_j=x_j}$$

Q-operator method

$$SP_\lambda(x_1, \dots, x_L) = \prod_{i=1}^L q_\lambda(x_i) P_\lambda(1, \dots, 1)$$

Construct operator \mathcal{A}_i :

$$\begin{aligned} \mathcal{A}_i P_\lambda(x_1, \dots, x_i, 1, \dots, 1) &= q_\lambda(x_i) P_\lambda(x_1, \dots, x_{i-1}, 1, \dots, 1) \\ &= (Q_z P_\lambda(x_1, \dots, x_L)) \Big|_{x_i, \dots, x_L=1} \Big|_{z=x_i} \end{aligned}$$

Separating operator is a chain of \mathcal{A}_i

$$S = \mathcal{A}_1 \dots \mathcal{A}_L$$

Q-operator method

Can also construct

$$S_k = \mathcal{A}_1 \dots \mathcal{A}_k$$

which acts as

$$S_k P_\lambda(x_1, \dots, x_k, 1, \dots, 1) = \prod_{i=1}^k q_\lambda(x_i) P_\lambda(1, \dots, 1)$$

Symplectic character

Let P_λ be the Symplectic character χ_λ .

$$\chi_\lambda(x_1, \dots, x_L) = \frac{a_\mu(x_1, \dots, x_L)}{a_\delta(x_1, \dots, x_L)},$$

$$a_\mu(x_1, \dots, x_L) = \det \left[x_i^{\mu_j} - x_i^{-\mu_j} \right]$$

$$\delta = (\dots, 3, 2, 1), \quad \mu = \lambda + \delta$$

Symplectic character

'Restricted polynomial'

$$\chi_\lambda(x_1, \dots, x_k, 1, \dots, 1) = \frac{a_\mu^{(k+1)}(x_1, \dots, x_k)}{a_\delta^{(k+1)}(x_1, \dots, x_k)},$$

$$a_\mu^{(k+1)}(x_1, \dots, x_k) = \det \begin{bmatrix} [x_i^{\mu_j} - x_i^{-\mu_j}]_{i \leq k} \\ [\mu_j^{2(L-i)+1}]_{i \geq k+1} \end{bmatrix}.$$

$a_\mu^{(k)}$ is proportional to a_μ ;

$$\lim_{x_k, \dots, x_L \rightarrow 1} a_\mu(x_1, \dots, x_L) = c_k a_\mu^{(k)}(x_1, \dots, x_{k-1})$$

Hamiltonian

The Hamiltonian

$$H_j \chi_\lambda(x_1, \dots, x_L) = h_j(\lambda) \chi_\lambda(x_1, \dots, x_L)$$

is of the form

$$H_j = \frac{1}{a_\delta(x_1, \dots, x_L)} \left[\dots \right]_j a_\delta(x_1, \dots, x_L)$$

$[\dots]_j$ can be found by noting

$$\left(x_i \frac{\partial}{\partial x_i} \right)^2 (x_i^{\mu_j} - x_i^{-\mu_j}) = \mu_j^2 (x_i^{\mu_j} - x_i^{-\mu_j})$$

Hamiltonian

The Hamiltonian

$$H_j \chi_\lambda(x_1, \dots, x_L) = h_j(\lambda) \chi_\lambda(x_1, \dots, x_L)$$

is of the form

$$H_j = \frac{1}{a_\delta(x_1, \dots, x_L)} \left[\dots \right]_j a_\delta(x_1, \dots, x_L)$$

Solution:

$$[\dots]_j = e_j \left(\left(x_1 \frac{\partial}{\partial x_1} \right)^2, \dots, \left(x_L \frac{\partial}{\partial x_L} \right)^2 \right)$$
$$h_j(\lambda) = e_j(\mu_1, \dots, \mu_L)$$

$$Q_z \chi_\lambda(x_1, \dots, x_L) = q_\lambda(z) \chi_\lambda(x_1, \dots, x_L)$$

The eigenvalue q_λ is related to χ_λ

$$\begin{aligned} q_\lambda(z) &= \frac{\chi_\lambda(z, 1, \dots, 1)}{\chi_\lambda(1, \dots, 1)} \\ &= \frac{a_\mu^{(2)}(z) a_\delta^{(1)}}{a_\delta^{(2)}(z) a_\mu^{(1)}} \end{aligned}$$

Q-operator

$$Q_z \chi_\lambda(x_1, \dots, x_L) = q_\lambda(z) \chi_\lambda(x_1, \dots, x_L)$$

Solution:

$$(Q_z f)(\underline{x}) = \frac{1}{a_\delta(\underline{x})} \int_1^z \frac{dw}{w} \int_{\mathcal{D}} d\mathbf{Y} a_\delta(\underline{y}) f(\underline{y})$$

The proof is equivalent to the proof of

$$\int_1^z \frac{dw}{w} \int_{\mathcal{D}} d\mathbf{Y} a_\mu(\underline{y}) = q_\lambda(z) a_\mu(\underline{x})$$

Double integrals: $\int \frac{dt_i}{t_i} \int \frac{dy_i}{y_i}, \quad x_i \leq \frac{y_i}{t_i} \leq x_{i+1}$

Operator \mathcal{A}_k

$$\mathcal{A}_k \chi_\lambda(x_1, \dots, x_k, 1, \dots, 1) = q_\lambda(x_k) \chi_\lambda(x_1, \dots, x_{k-1}, 1, \dots, 1)$$

Solution:

$$(\mathcal{A}_k f)(x_1, \dots, x_k) = \frac{1}{a_\delta^{(k)}(x_1, \dots, x_{k-1})} \int_1^{x_k} \frac{dw}{w} \int_{\mathcal{D}'} d\mathbf{Y}' a_\delta^{(k+1)}(y_1, \dots, y_k) f(y_1, \dots, y_k)$$

Separating operator

Can now construct the separating operator

$$\mathcal{S} = \mathcal{A}_1 \dots \mathcal{A}_L,$$

as well as

$$\mathcal{S}_k = \mathcal{A}_1 \dots \mathcal{A}_k$$

which acts on restricted χ_λ :

$$\mathcal{S}_k \chi_\lambda(x_1, \dots, x_k, 1, \dots, 1) = \prod_{i=1}^k q_\lambda(x_i) \chi_\lambda(1, \dots, 1).$$

Inverse separating operator

$$\mathcal{S}^{-1} \prod_{i=1}^L q_{\lambda}(x_i) = \frac{\chi_{\lambda}(x_1, \dots, x_L)}{\chi_{\lambda}(1, \dots, 1)}$$

Solution:

$$\mathcal{S}^{-1} = \frac{a_{\delta}^{(1)}}{a_{\delta}(x_1, \dots, x_L)} \det \left[\left(x_i \frac{\partial}{\partial x_i} \right)^{2(L-j)} \right] \prod_{i=1}^L \frac{a_{\delta}^{(2)}(x_i)}{a_{\delta}^{(1)}}$$

The proof of this is equivalent to

$$\det \left[\left(x_i \frac{\partial}{\partial x_i} \right)^{2(L-j)} \right] \prod_{i=1}^L \frac{a_{\mu}^{(2)}(x_i)}{a_{\mu}^{(1)}} = \frac{a_{\mu}(x_1, \dots, x_L)}{a_{\mu}^{(1)}}$$

Inverse separating operator

$$\mathcal{S}_k^{-1} \prod_{i=1}^k q_\lambda(x_i) = \frac{\chi_\lambda(x_1, \dots, x_k, 1, \dots, 1)}{\chi_\lambda(1, \dots, 1)}$$

Solution:

$$\mathcal{S}_k^{-1} = \frac{a_\delta^{(1)}}{a_\delta(x_1, \dots, x_k)} \det_k \left[\left(x_i \frac{\partial}{\partial x_i} \right)^{2(k-j)} \right] \prod_{i=1}^k \frac{a_\delta^{(2)}(x_i)}{a_\delta^{(1)}}$$

Factorised Hamiltonian

It can be shown that q_λ satisfies the D.E.

$$W_x q_\lambda(x) = 0,$$

$$W_x = \prod_{n=1}^L \left(x \frac{\partial}{\partial x} + L \left(\frac{x+1}{x-1} \right) - \frac{2x}{x^2-1} \right)^2 - \mu_n^2.$$

Define $W_{i,j} = W_{x_i} + h_j(\lambda)$, so

$$\begin{aligned} W_{i,j} q_\lambda(x_i) &= h_j(\lambda) q_\lambda(x_i) \\ \Rightarrow W_{i,j} \prod_{n=1}^L q_\lambda(x_n) &= h_j(\lambda) \prod_{n=1}^L q_\lambda(x_n) \quad \forall i. \end{aligned}$$

Factorised Hamiltonian

$$W_{i,j} \prod_{n=1}^L q_{\lambda}(x_n) = h_j(\lambda) \prod_{n=1}^L q_{\lambda}(x_n) \quad \forall i$$

Therefore any linear combination of $W_{i,j}$

$$\tilde{H}_j = \sum_i c_i W_{i,j}$$

acts like SH_jS^{-1} , ie

$$\tilde{H}_j \prod_{n=1}^L q_{\lambda}(x_n) = h_j(\lambda) \prod_{n=1}^L q_{\lambda}(x_n).$$

\Rightarrow factorised version of the spectral problem.

Conclusion

- Found Q -operator, and \mathcal{A}_i operator
- From this, constructed separating operator \mathcal{S} for symplectic character χ_λ and \mathcal{S}_k for restricted symplectic character
- Found inverse separating operators \mathcal{S} and \mathcal{S}_k
- Constructed factorised Hamiltonian \tilde{H}_j from D.E. for q_λ .

Outlook:

- Asymptotics of $\chi_\lambda(\zeta_1, \zeta_2, 1, \dots, 1)$ ($L \rightarrow \infty$) can be obtained from asymptotics of $q_\lambda(\zeta)$ by using the inverse separating operator \mathcal{S}_2
- Separating operator for more general Jack polynomials of type BC , or other root systems?

Thank you for your attention

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