Schramm's left-passage probability in a finite system

Anita Ponsaing

Université de Genève

1 December, 2011

Stat Mech Meeting, University of Melbourne

Work done in collaboration with Yacine Ikhlef

Summary

Introduction to SLE

Left (or right) passage probability — Schramm's Formula

Left passage probability — Finite-size calculation

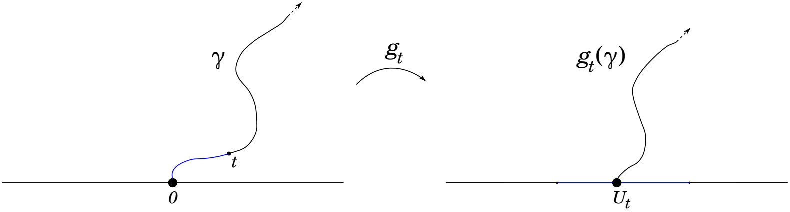
Numerical Results

Summary

Schramm–Loewner evolution (SLE), related to:

- Ising model
- Critical percolation
- Loop-erased Random Walks
- etc. . .

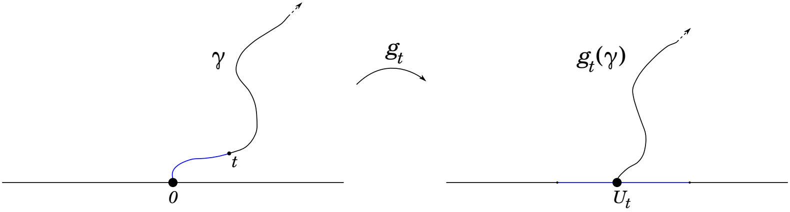
SLE Introduction



$$g_t(\gamma[0,t]) \subseteq \mathbb{R},$$

$$U_t = g_t(\gamma(t)).$$

SLE Introduction



$$g_t(\gamma[0,t]) \subseteq \mathbb{R},$$

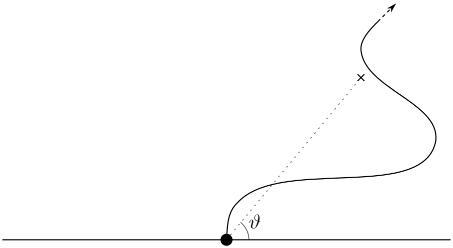
$$U_t = g_t(\gamma(t)).$$

Loewner: $$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - U_t}, \qquad \text{with } g_0(z) = z.$$

Schramm: Let $U_t = \sqrt{\kappa}\; B_t$ (Brownian Motion)

Schramm's Formula

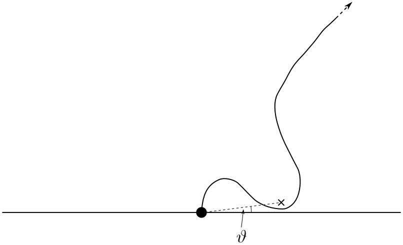
Schramm (2001): Right-passage probability $P(\theta)$



$$P(\theta) = \frac{1}{2} - \frac{\Gamma(4/\kappa)}{\sqrt{\pi}\ \Gamma(\frac{8-\kappa}{2\kappa})}\ \frac{1}{\tan\theta}\ {}_2F_1\left(\frac{1}{2}, \frac{4}{\kappa}, \frac{3}{2}, \frac{-1}{\tan^2\theta}\right).$$

Schramm's Formula

Schramm (2001): Right-passage probability $P(\theta)$

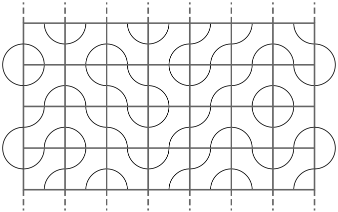


$\kappa = 6$, limit as $\theta \to 0$

$$P(\theta) \overset{\theta \to 0}{\sim} \frac{3\ \Gamma(2/3)}{\sqrt{\pi}\ \Gamma(1/6)} \theta^{1/3}.$$

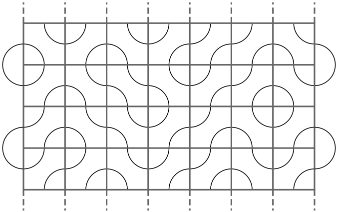
Finite-size calculation — O($n = 1$) loop model

Completely packed O($n = 1$) loop model with reflecting boundaries



Finite-size calculation — O($n = 1$) loop model

Completely packed O($n = 1$) loop model with reflecting boundaries

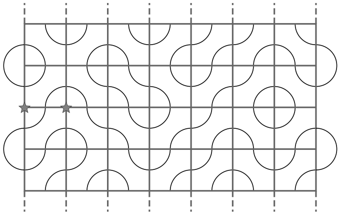


Conformal map: $\theta = \frac{\pi k}{L}$.

$$-\infty \mapsto 0; \qquad \mathrm{LB} \mapsto \mathbb{R}^+; \qquad \mathrm{RB} \mapsto \mathbb{R}^-;$$

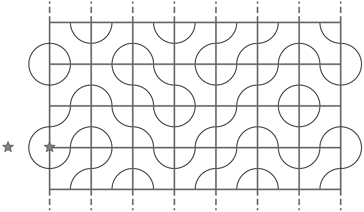
Left passage $\mapsto$ Right passage.

First site passage:

$X_1 =$ 

Finite-size calculation — O($n = 1$) loop model

Boundary passage:

$\tilde{X}_0 =$ 

Finite-size calculation — O($n = 1$) loop model

$$X_1 = \frac{\chi_{L-1}(z_2, \ldots, z_L)\chi_{L+1}(z_1, z_1, z_2, \ldots, z_L)}{Z_L^2}$$

where

$$\chi_L(\mathbf{z}) = \frac{\det \left| z_i^{\left(\left\lfloor \frac{L-j}{2} \right\rfloor + L - j + 1\right)} - z_i^{-\left(\left\lfloor \frac{L-j}{2} \right\rfloor + L - j + 1\right)} \right|}{\det \left| z_i^{(L-j+1)} - z_i^{-(L-j+1)} \right|},$$

and

$$Z_L = \chi_L(\mathbf{z})$$

Finite-size calculation — O($n = 1$) loop model

$$\tilde{X}_0 = \frac{1}{A(\mathbf{z})} \frac{\chi_{L+1}(w, z_1, \ldots, z_L)\chi_{L+1}(q/w, z_1, \ldots, z_L)}{Z_L^2}$$

where

$$\chi_L(\mathbf{z}) = \frac{\det \left| z_i^{\left(\left\lfloor \frac{L-j}{2} \right\rfloor + L - j + 1\right)} - z_i^{-\left(\left\lfloor \frac{L-j}{2} \right\rfloor + L - j + 1\right)} \right|}{\det \left| z_i^{(L-j+1)} - z_i^{-(L-j+1)} \right|},$$

and

$$Z_L = \chi_L(\mathbf{z})$$

Finite-size calculation — O($n = 1$) loop model

$$X_1 \overset{L\to\infty}{\sim} C\ L^{-1/3}$$

$$\tilde{X}_0 \overset{L\to\infty}{\sim} \tilde{C}\ L^{-1/3} \quad \text{(probably)}$$

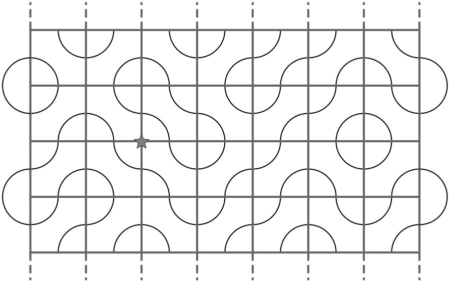
Recall $\theta = \pi k/L$, and

$$P(\theta) \overset{\theta\to 0}{\sim} \frac{3\ \Gamma(2/3)}{\sqrt{\pi}\ \Gamma(1/6)} \theta^{1/3}.$$

But $C \neq \frac{3\sqrt{\pi}\ \Gamma(2/3)}{\Gamma(1/6)}$!

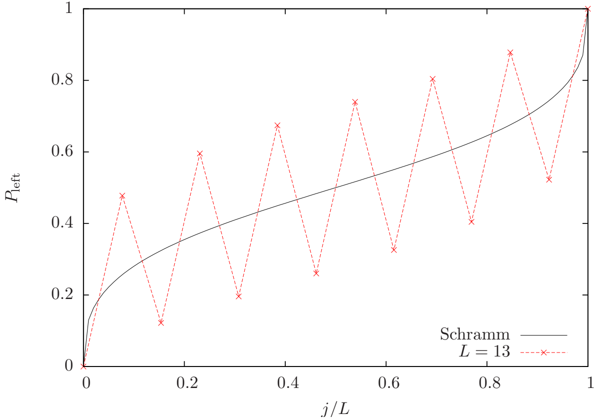
Numerics

Preamble: $P(\text{left of } k) = X_1 - X_2 + \ldots + (-1)^k X_k$.



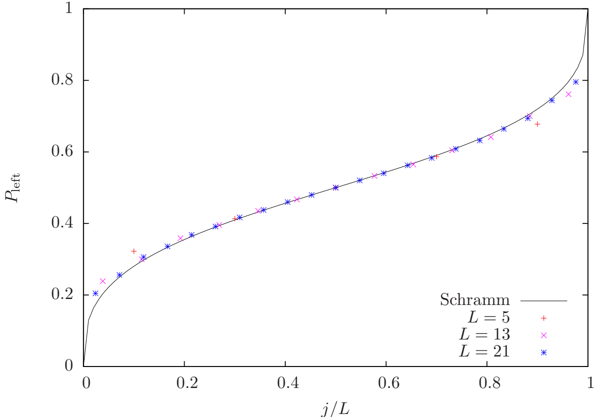
Numerics

Schramm's formula and finite left passage P_k:



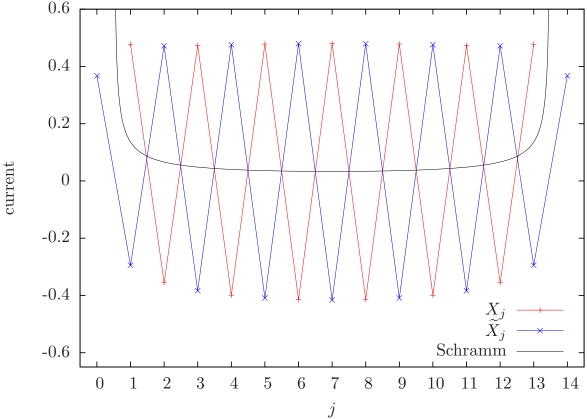
Numerics

Schramm's formula and average of P_k and P_{k+1}:



Numerics

Derivative of Schramm's formula and single-site passage (X and $\tilde{X}$):



Conclusion

- Finite-size calculation for first-site and boundary passage
- Asymptotic expression for first-site passage (critical exponent matches)
- Good evidence for the relationship between $O(n = 1)$ loop model (accounting for parity) and SLE

To Do:

- Asymptotic expression for the boundary passage
- Exact formulas for each site passage (then asymptotics)

Conclusion

- Finite-size calculation for first-site and boundary passage
- Asymptotic expression for first-site passage (critical exponent matches)
- Good evidence for the relationship between $O(n = 1)$ loop model (accounting for parity) and SLE

To Do:

- Asymptotic expression for the boundary passage
- Exact formulas for each site passage (then asymptotics)

Thank you