Edge currents in the O($n = 1$) loop model

Anita Ponsaing

University of Melbourne

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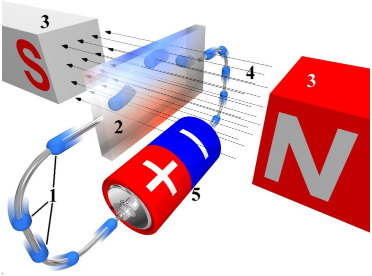
Work done in collaboration with Jan de Gier and Bernard Nienhuis

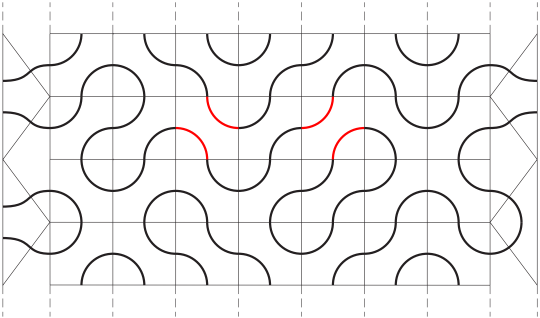
The quantum spin Hall effect

The (non-quantum, non-spin) Hall Effect:

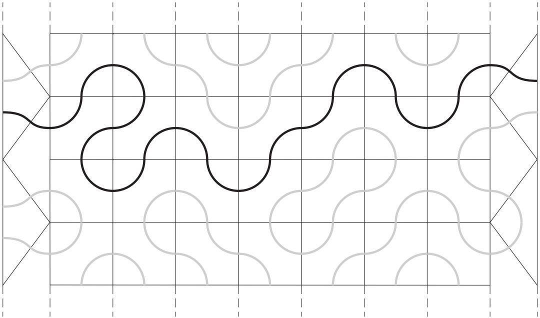


Source: Wikipedia

The lattice (Example: $L = 7$)

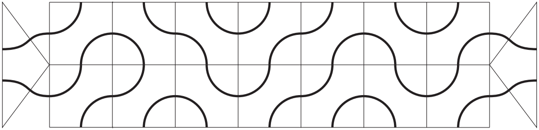


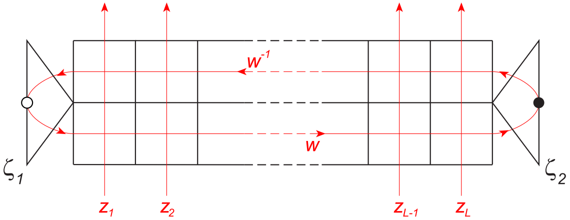
The lattice (Example: $L = 7$)



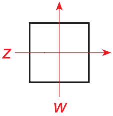
The lattice (Example: $L = 7$)

One of the terms of the transfer matrix:



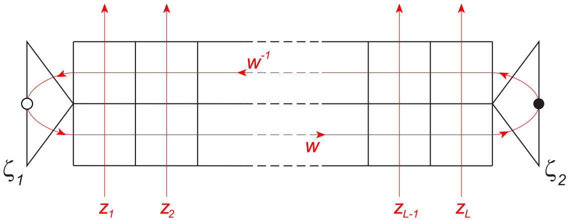
$T(w, \mathbf{z}) =$ 

$$\mathbf{z} = z_1, \ldots, z_L$$

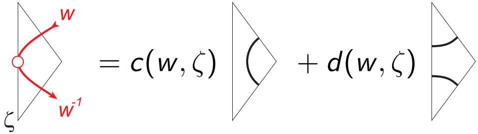


$= a(z/w)$  $+ b(z/w)$ 

Transfer Matrix

$T(w, \mathbf{z}) =$ 

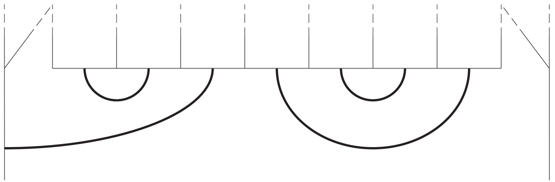
$$\mathbf{z} = z_1, \ldots, z_L$$



The eigenvector

$$T|\Psi\rangle = |\Psi\rangle$$
$$|\Psi\rangle = \sum_{\alpha} \psi_\alpha |\alpha\rangle$$

Example α:



$$\psi_\alpha = \psi_\alpha(\zeta_1, \zeta_2; \mathbf{z}) = \text{probability of connectivity } \alpha$$

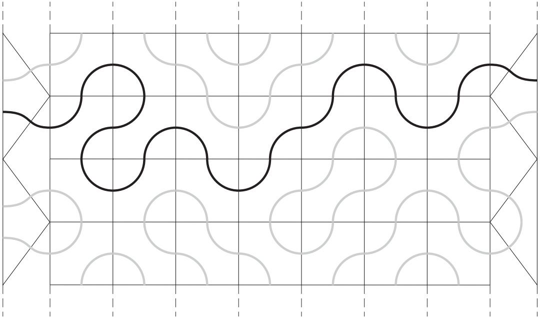
The partition function

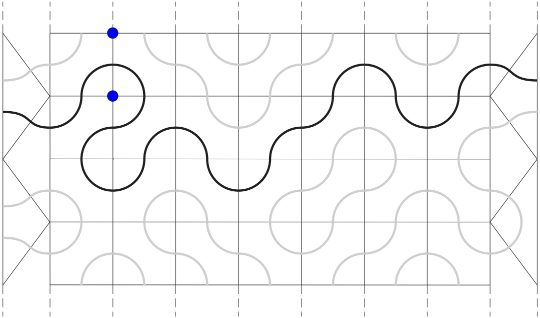
Partition function:

$$Z_L(\mathbf{z}) = \sum_{configurations} \prod_{faces} w_{face}$$

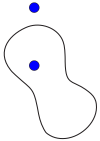
$$= \sqrt{\langle \Psi(\mathbf{z}) | \Psi(\mathbf{z}) \rangle}$$

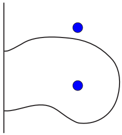
$$= \tau(\mathbf{z})\, \tau(\mathbf{z}, \zeta_1)\, \tau(\mathbf{z}, \zeta_2)\, \tau(\mathbf{z}, \zeta_1, \zeta_2)$$



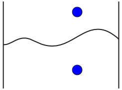


weight 0:

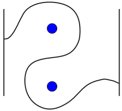




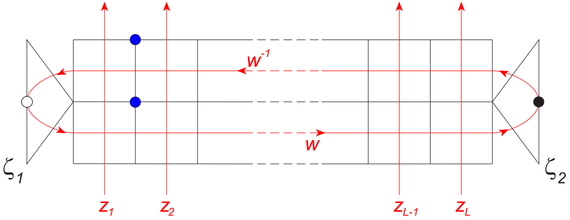
weight 1:



weight -1:



$\hat{F}_L^{(2,vert.)} =$



Solution

$$F_L^{(k)} = \frac{\langle \Psi(\mathbf{z}) | \hat{F}^{(k)} | \Psi(\mathbf{z}) \rangle}{\langle \Psi(\mathbf{z}) | \Psi(\mathbf{z}) \rangle}$$

$$F_L^{(k,horiz.)} = z_k \frac{\partial}{\partial z_k} \; u_L(\zeta_1, \zeta_2; z_1, \dots, z_L)$$

$$F_L^{(k,vert.)} = w \; \frac{\partial}{\partial w} \; u_{L+2}(\zeta_1, \zeta_2; z_1, \dots, z_L, v/q, w) \Big|_{v=w}$$

where

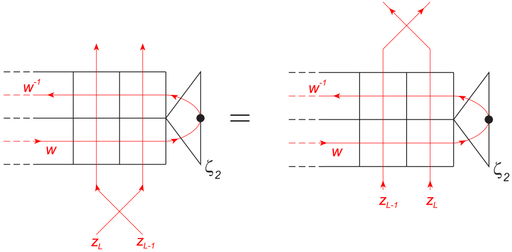
$$u_L = (-1)^L i \frac{\sqrt{3}}{2} \log \left[\frac{\tau(\zeta_1, \mathbf{z}) \tau(\zeta_2, \mathbf{z})}{\tau(\zeta_1, \zeta_2, \mathbf{z}) \tau(\mathbf{z})} \right]$$

Summary of proof

Induction:

- Prove solution for $L = 2$ and $L = 3$
- Prove that the solution satisfies all recursions *(necessary condition)*
- Prove that there are enough recursions to fix the solution *(sufficient condition)*

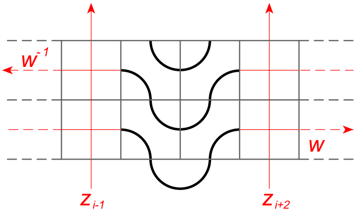
Symmetries



Leads to:

- qKZ equation
- Z_L symmetric in z_is
- $F_L^{(k,horiz.)}$ symmetric in z_is, $i \neq k$
- $F_L^{(k,vert.)}$ symmetric in z_is

Recursions



$$Z_L\big|_{z_i=qz_j} \propto Z_{L-2}$$

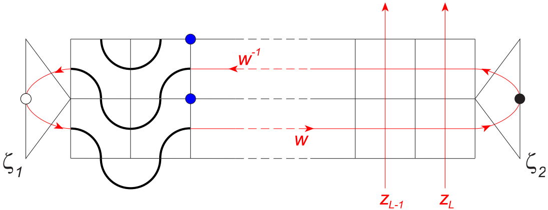
$$Z_L\big|_{z_i=q\zeta_1} \propto Z_{L-1}$$

$$Z_L\big|_{z_i=q/\zeta_2} \propto Z_{L-1}$$

where q is a special parameter which is set to $e^{2\pi i/3}$.

Example recursion

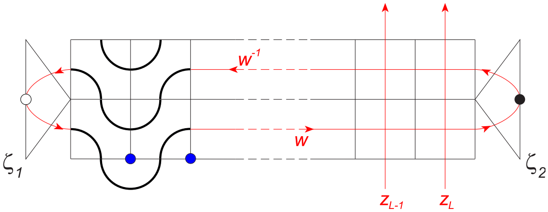
When $z_2 = qz_1$,



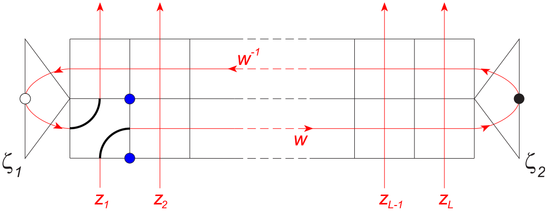
$$F_L \propto F_{L-2}$$

More example recursions

$F_L^{(horiz.)} \propto F_{L-2}^{(vert.)}$:



$F_L^{(vert.)} \propto F_L^{(horiz.)}$:



Conclusion

arXiv:1004.4037

- Set out to find probability F_k of a path touching both boundaries
- Outlined the proof of the forms of F_k

Outlook:

- Interesting connection to Toda lattice wavefunctions
- Asymptotics ($L \to \infty$)

Thank you for your attention