

The two-boundary Brauer loop model

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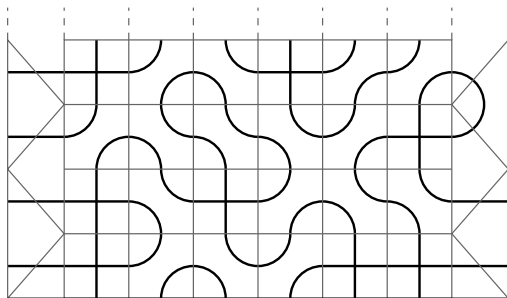
Stony Brook

Work-in-progress in collaboration with Paul Zinn-Justin

Introduction

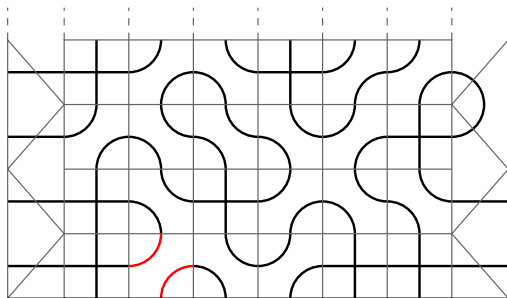
Introduction: The Brauer model

Loop model on a semi-infinite lattice:



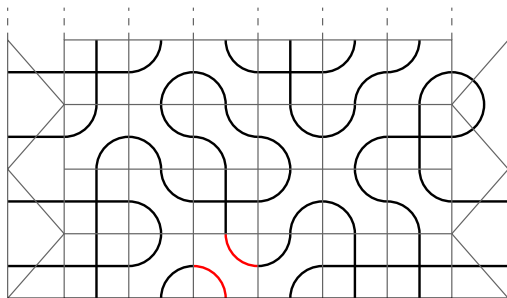
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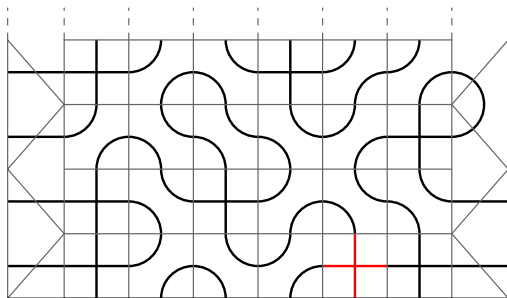
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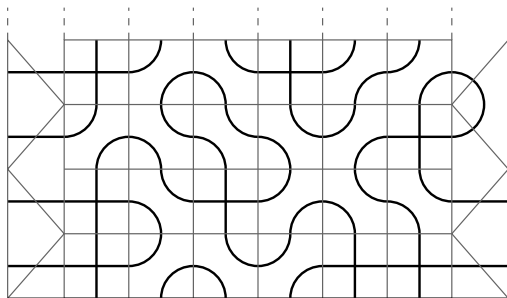
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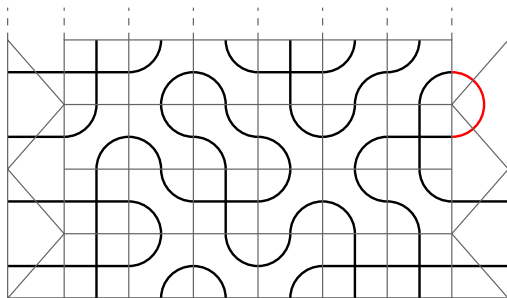
Loop model on a semi-infinite lattice:



$$R = a \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} + b \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} + c \begin{array}{|c|} \hline \text{---} \\ \hline \end{array}$$

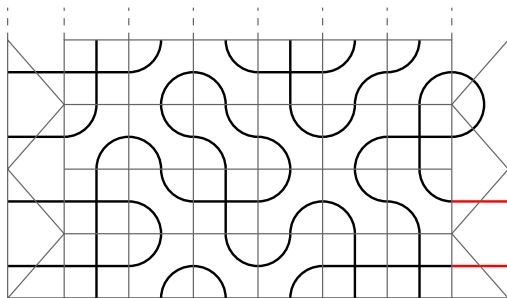
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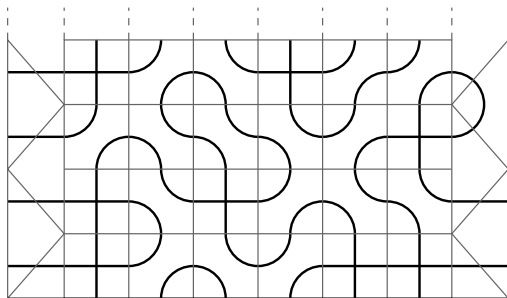
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Loop model on a semi-infinite lattice:



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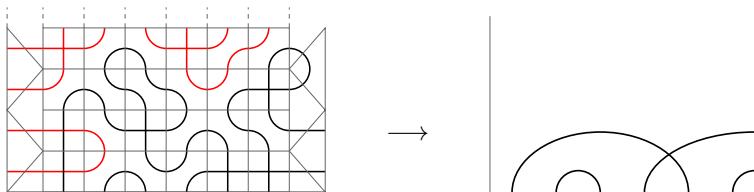
Loop model on a semi-infinite lattice:



$$K = k_1 \left\langle \begin{array}{c} \diagup \\ \diagdown \end{array} \right\rangle + k_2 \left\langle \begin{array}{c} \diagdown \\ \diagup \end{array} \right\rangle$$

Link patterns

Sites can be connected to each other or to the boundary.
Ignore closed loops and loops connected only to the boundaries:



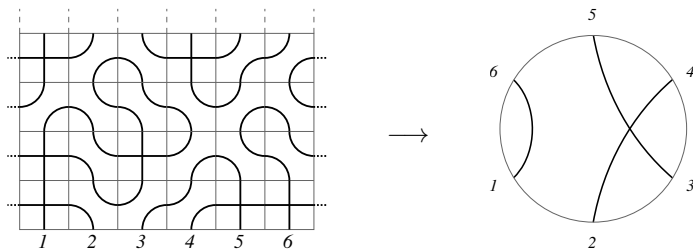
“Probability” vector, with α a link pattern:

$$|\Psi\rangle = \sum_{\alpha} \psi_{\alpha} |\alpha\rangle$$

Sum rule:

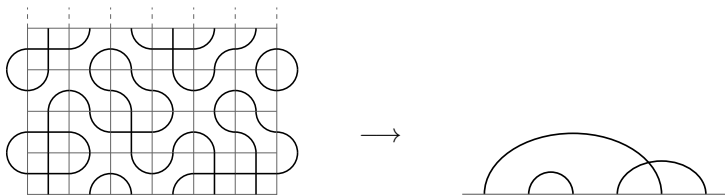
$$Z_L = \sum_{\alpha} \psi_{\alpha}$$

Other boundary conditions: Periodic



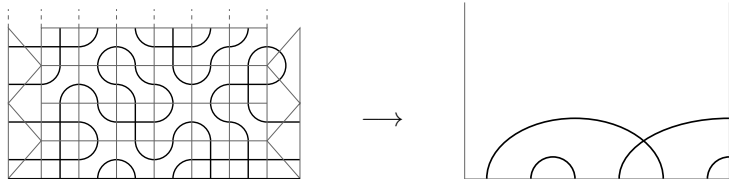
- ▶ [de Gier & Nienhuis, 2005]: Conjectured agreement between probabilities of link patterns and degrees of the 'upper-upper' algebraic variety from [Knutson, 2003].
- ▶ [Di Francesco & Zinn-Justin, 2006]: Refinement of conjecture, calculations of ψ_α , Z_L and sum of **permutation-type** components.
- ▶ [Knutson & Zinn-Justin, 2007]: Proof of conjecture, involving permutation-type components.

Other boundary conditions: Reflecting



- ▶ [Di Francesco, 2005]: Calculations of ψ_α , Z_L and sum of permutation-type components.

This talk: Open boundaries



- ▶ Calculations of (some) ψ_α , Z_L and sum of permutation-type components. (Almost complete proof)

Details

R Matrix

Introduce inhomogeneities.

$$R(w - u) = a(w - u) \begin{array}{|c|} \hline \text{↖ ↗} \\ \hline \end{array} + b(w - u) \begin{array}{|c|} \hline \text{↗ ↖} \\ \hline \end{array} + c(w - u) \begin{array}{|c|} \hline \text{+} \\ \hline \end{array}$$
$$= w \begin{array}{|c|} \hline \text{+} \\ \hline \end{array} \begin{array}{|c|} \hline \text{+} \\ \hline \end{array} \begin{array}{|c|} \hline \text{+} \\ \hline \end{array}$$

u

Probability of configurations on a face:

$$a(z) = \frac{2(z-1)}{(z+1)(z-2)}, \quad b(z) = \frac{-2z}{(z+1)(z-2)}, \quad c(z) = \frac{z(z-1)}{(z+1)(z-2)}$$

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$$= w \begin{array}{|c|c|} \hline \uparrow & \downarrow \\ \hline \hline \hline \end{array} \begin{array}{|c|} \hline \rightarrow \\ \hline \end{array} \begin{array}{|c|} \hline \downarrow \\ \hline \end{array} \begin{array}{|c|} \hline \rightarrow \\ \hline \end{array} \\ u$$

Probability of configurations on a face:

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Chosen so that **Yang-Baxter equation** holds:

The diagram illustrates the Yang-Baxter equation for a square face. On the left, a square face has two diamonds on its left side. Two horizontal lines with arrows enter from the left and exit to the right. The top line has an arrow pointing right, and the bottom line has an arrow pointing right. The top line passes through the top diamond, and the bottom line passes through the bottom diamond. On the right, the same square face has two diamonds on its right side. The two horizontal lines with arrows enter from the left and exit to the right. The top line has an arrow pointing right, and the bottom line has an arrow pointing right. The top line passes through the top diamond, and the bottom line passes through the bottom diamond. The two configurations are shown to be equivalent with an equals sign between them.

K Matrices

$$K_0(w) = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \curvearrowright \\ \text{---} \\ \curvearrowleft \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = k_1(1-w) \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \curvearrowright \\ \text{---} \end{array} + k_2(1-w) \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \curvearrowleft \\ \text{---} \end{array}$$

$$K_L(w) = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \curvearrowleft \\ \text{---} \\ \curvearrowright \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = k_1(w) \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \curvearrowleft \\ \text{---} \end{array} + k_2(w) \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \curvearrowright \\ \text{---} \end{array}$$

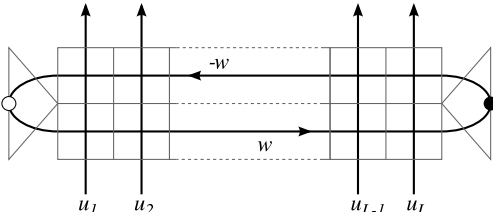
Inhomogeneous probabilities:

$$k_1(w) = \frac{1-2w}{1+2w}, \quad k_2(w) = \frac{4w}{1+2w}$$

Chosen so **boundary YBE** holds.

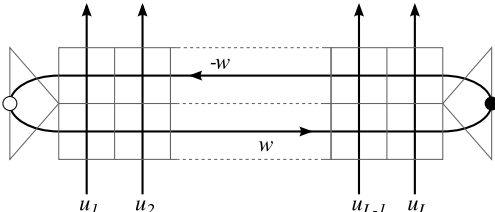
Transfer Matrix

Probabilities of configurations on two rows of the lattice:

$$T(w; u_1, \dots, u_L) =$$


Transfer Matrix

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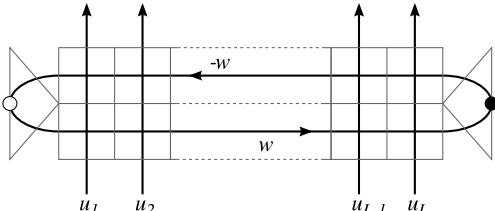
The diagram illustrates the transfer matrix T for a two-row lattice. It consists of a grid of two rows and L columns. The left boundary is a white circle, and the right boundary is a black circle. Two horizontal arrows represent weights w and $-w$. Vertical arrows represent weights u_1, u_2, u_{L-1}, u_L .

Ground state:

$$|\Psi(u_1, \dots, u_L)\rangle = \sum_{\alpha} \psi_{\alpha}(u_1, \dots, u_L) |\alpha\rangle$$

Transfer Matrix

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The diagram illustrates the transfer matrix T for a two-row lattice. It consists of a grid of two rows and L columns. The left boundary is a white circle, and the right boundary is a black circle. Two horizontal arrows represent weights w and $-w$. Vertical arrows represent site variables u_1, u_2, u_{L-1}, u_L .

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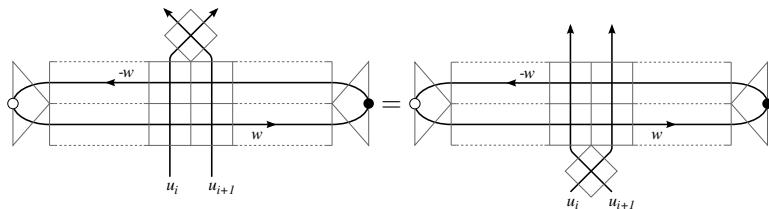
$$|\Psi(u_1, \dots, u_L)\rangle = \sum_{\alpha} \psi_{\alpha}(u_1, \dots, u_L) |\alpha\rangle$$

Eigenvalue of 1:

$$T(w; u_1, \dots, u_L) |\Psi(u_1, \dots, u_L)\rangle = |\Psi(u_1, \dots, u_L)\rangle$$

Interlacing relation (bulk)

YBE implies:

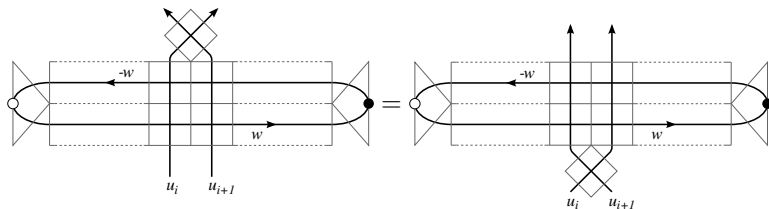


Equivalently,

$$R(u_i - u_{i+1})T(w, \dots, u_i, u_{i+1}, \dots) = T(w, \dots, u_{i+1}, u_i, \dots)R(u_i - u_{i+1})$$

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Leading to **quantum Knizhnik–Zamolodchikov (qKZ) equation**:

$$R(u_i - u_{i+1})|\Psi(\dots, u_i, u_{i+1}, \dots)\rangle = |\Psi(\dots, u_{i+1}, u_i, \dots)\rangle$$

q KZ equation (bulk)

$$R(u_i - u_{i+1})|\Psi(\dots, u_i, u_{i+1}, \dots)\rangle = |\Psi(\dots, u_{i+1}, u_i, \dots)\rangle$$

R in terms of Brauer algebra generators:

$$\begin{aligned} R(u_i - u_{i+1}) &= a(u_i - u_{i+1}) \begin{array}{c} \dots \\ \dots \\ \dots \end{array} \begin{array}{|c|c|} \hline i & i+1 \\ \hline \hline \end{array} \begin{array}{c} \dots \\ \dots \\ \dots \end{array} + b(u_i - u_{i+1}) \begin{array}{c} \dots \\ \dots \\ \dots \end{array} \begin{array}{c} \overset{i}{\cup} \quad \overset{i+1}{\cup} \\ \hline \cup \quad \cup \end{array} \begin{array}{c} \dots \\ \dots \\ \dots \end{array} + c(u_i - u_{i+1}) \begin{array}{c} \dots \\ \dots \\ \dots \end{array} \begin{array}{|c|c|} \hline i & i+1 \\ \hline \diagdown \quad \diagup \\ \hline \end{array} \begin{array}{c} \dots \\ \dots \\ \dots \end{array} \\ &= a_i + b_i e_i + c_i f_i \end{aligned}$$

e_i & f_i act on $|\alpha\rangle$, so q KZ equation becomes a relationship between the components of $|\Psi\rangle$.

q KZ equation (bulk)

$$(a_i + b_i e_i + c_i f_i) |\Psi\rangle = \pi_i |\Psi\rangle \quad (= |\Psi(u_{i+1}, u_i)\rangle)$$

Three cases for $|\alpha\rangle$:

- ▶ $f_i |\alpha\rangle = |\alpha\rangle$ & α has no loop from i to $i + 1$,

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- ▶ $f_i |\alpha\rangle = |\beta\rangle \neq |\alpha\rangle$, \Rightarrow α has no loop from i to $i+1$,

$$a_i \psi_\alpha + c_i \psi_\beta = \pi_i \psi_\alpha$$

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q KZ equation (bulk)

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$$(a_i + c_i) \psi_\alpha = \pi_i \psi_\alpha$$

- ▶ $f_i |\alpha\rangle = |\beta\rangle \neq |\alpha\rangle$, \Rightarrow α has no loop from i to $i + 1$,

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$$a_i \psi_\beta + c_i \psi_\alpha = \pi_i \psi_\beta$$

- ▶ α has a loop from i to $i + 1$, \Rightarrow $f_i |\alpha\rangle = |\alpha\rangle$,

$$(a_i + c_i) \psi_\alpha + b_i \sum_{\gamma: e_i |\gamma\rangle = |\alpha\rangle} \psi_\gamma = \pi_i \psi_\alpha$$

q KZ equation (bulk)

First case: $f_i|\alpha\rangle = |\alpha\rangle$ & α has no loop from i to $i+1$,

$$(a_i + c_i)\psi_\alpha = \pi_i\psi_\alpha$$

More detail:

$$\begin{aligned}\psi_\alpha(u_{i+1}, u_i) &= (a_i + c_i)\psi_\alpha(u_i, u_{i+1}) \\ &= \frac{(u_{i+1} - u_i + 1)(u_{i+1} - u_i - 2)}{(u_i - u_{i+1} + 1)(u_i - u_{i+1} - 2)}\psi_\alpha(u_i, u_{i+1})\end{aligned}$$

$$\begin{aligned}\Rightarrow \psi_\alpha(u_i, u_{i+1}) &= (u_i - u_{i+1} + 1)(u_i - u_{i+1} - 2)S^{\{i, i+1\}}(u_i, u_{i+1}) \\ &= r(u_i - u_{i+1})S^{\{i, i+1\}}(u_i, u_{i+1})\end{aligned}$$

Results

Special Components

q KZ directly implies:

$$\begin{aligned} \psi_{\underbrace{L+1, \dots, L+1}_k, \underbrace{0, \dots, 0}_{L-k}} &= \prod_{i=1}^k (1 - 2u_i) \prod_{i=k+1}^L (1 + 2u_i) \\ &\times \prod_{1 \leq i < j \leq k} r(u_i - u_j) r(-u_i - u_j) \prod_{k+1 \leq i < j \leq L} r(u_i - u_j) r(u_i + u_j) \\ &\times \prod_{i=1}^k \prod_{j=k+1}^L [(u_i - u_j + 1)(-u_i - u_j + 1) (u_i + u_j + 1)(-u_i + u_j + 1)] \\ &\times S^{\{0, \dots, k\}, \{k+1, \dots, L\}}(u_1, \dots, u_L), \end{aligned}$$

Conjecture (bound on degree): $S = 1$.

Recursion (bulk)

Can also show (no details given here):

$$|\Psi_L(\dots, u_i, u_i + 1, \dots)\rangle = p(u_i; \dots, \hat{u}_i, \hat{u}_{i+1}, \dots) |\Psi_{L-2}(\dots, \hat{u}_i, \hat{u}_{i+1}, \dots)\rangle$$

Implying

$$Z_L(\dots, u_i, u_i + 1, \dots) = p(u_i; \dots, \hat{u}_i, \hat{u}_{i+1}, \dots) Z_{L-2}(\dots, \hat{u}_i, \hat{u}_{i+1}, \dots).$$

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With the conjectured bound on the degree, we have

$$p(u_i; \dots) = (1 - 2u_i)^2 (1 + 2(u_i + 1))^2 \prod_{j \neq i, i+1} r(u_j - u_i)^2 r(u_j + u_i + 1)^2$$

This recursion is useful for proving the sum rule (next slide).

Sum Rule

With the conjectured bound on the degree,

$$Z_{2n} = \prod_{1 \leq i < j \leq 2n} b(u_i, u_j)^2 \det \left[\frac{(-5 + 2u_i^2 + u_j^2)}{b(u_i, u_j)} \right]_{1 \leq i, j \leq 2n}$$

$$Z_{2n-1} = 2 \prod_{1 \leq i < j \leq 2n-1} b(u_i, u_j)^2 \\ \times \det \left[\begin{array}{cc} \left[\frac{(-5 + 2u_i^2 + 2u_j^2)}{b(u_i, u_j)} \right]_{1 \leq i, j \leq 2n-1} & [-1]_{1 \leq j \leq 2n-1} \\ [1]_{1 \leq i \leq 2n-1} & 0 \end{array} \right]$$

where

$$b(u_i, u_j) = \frac{(1 - (u_i - u_j)^2)(1 - (u_i + u_j)^2)}{u_i^2 - u_j^2}$$

$W_{2n} = \sum \psi_\alpha$ where α is a permutation between sites $1, \dots, n$ and $n+1, \dots, 2n$.

$$W_{2n}(u_1, \dots, u_{2n}) = \sqrt{Z_{2n}} \prod_{i=1}^n (1 - 2u_i)(1 + 2u_{n+i}) \\ \times \prod_{1 \leq i < j \leq n} r(u_i - u_j)r(-u_i - u_j)r(u_{n+i} - u_{n+j})r(u_{n+i} + u_{n+j})$$

Conclusion

- ▶ Need proof of degree
- ▶ Still working on proper proof for W_{2n}
- ▶ Working on model with general closed loop weight, $n = 2$ corresponds to non-crossing case
- ▶ Looking for connection to algebraic varieties

Thank you