The Brauer loop scheme with boundaries

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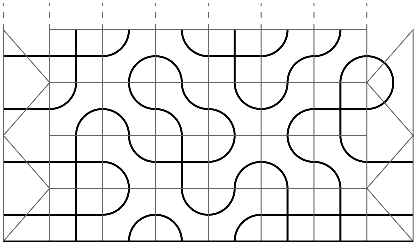
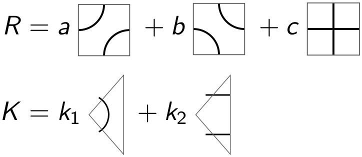
Brauer loop model

The Brauer model

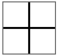
Loop model on a semi-infinite lattice. Closed loops are ignored ($\tau = 1$), focus on connectivities.

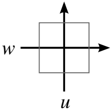




R Matrix

Introduce inhomogeneities.

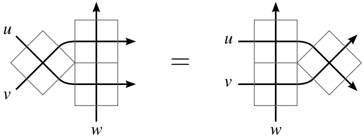
$$R(w-u) = a(w-u)$$  $$+\, b(w-u)$$  $$+\, c(w-u)$$ 

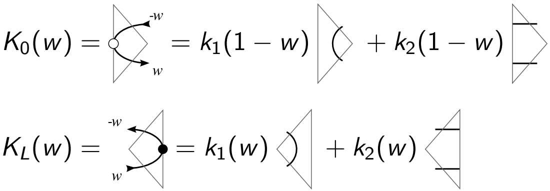
$$=$$ 

Probability of configurations on a face:

$$a(z) = \frac{2A(A-z)}{(A+z)(2A-z)}, \quad b(z) = \frac{2Az}{(A+z)(2A-z)}, \quad c(z) = \frac{z(A-z)}{(A+z)(2A-z)}$$

Chosen so that Yang–Baxter equation holds:





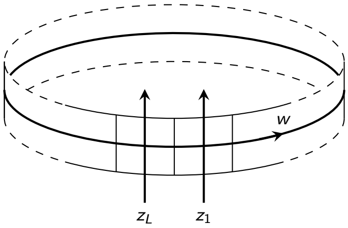
Inhomogeneous probabilities (depending on boundary conditions):

$$k_1(w) = \frac{1-2w}{1+2w}, \quad k_2(w) = \frac{4w}{1+2w}$$

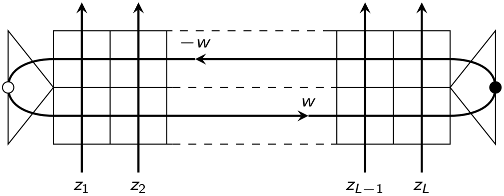
Chosen so that boundary YBE holds.

Transfer Matrix

Probabilities of configurations on one row of the lattice (periodic):

$T(w; z_1, \ldots, z_L) =$ 

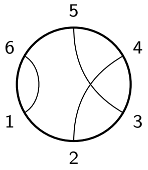
or two rows (all other BCs):

$T(w; z_1, \ldots, z_L) =$ 

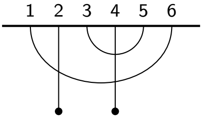
Link patterns

LP_L^a is the set of all link patterns with boundary condition a:

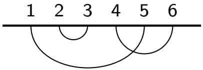
Periodic



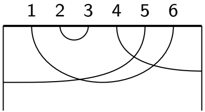
Identified



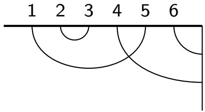
Closed



Open



Mixed



Ground state eigenvector

Steady state of stochastic process is ground state of transfer matrix

$$|\Psi(u_1, \dots, u_L)\rangle = \sum_{\alpha \in \mathrm{LP}_L} \psi_\alpha(u_1, \dots, u_L)|\alpha\rangle$$

with eigenvalue of 1:

$$T(w; u_1, \dots, u_L)\ |\Psi(u_1, \dots, u_L)\rangle = |\Psi(u_1, \dots, u_L)\rangle$$

Sum rule (eigenvector normalization):

$$Z_L = \sum_\alpha \psi_\alpha$$

qKZ equation

Eigenvector equation leads via Yang–Baxter to quantum Knizhnik–Zamolodchikov equation:

$$\check{R}(u_i - u_{i+1})|\Psi(\dots, u_i, u_{i+1}, \dots)\rangle = |\Psi(\dots, u_{i+1}, u_i, \dots)\rangle$$
$$\check{K}_0(-u_1)|\Psi(u_1, \dots)\rangle = |\Psi(-u_1, \dots)\rangle$$
$$\check{K}_L(u_L)|\Psi(\dots, u_L)\rangle = |\Psi(\dots, -u_L)\rangle$$

where $\check{R}$ is tilted version of R.

The LHS acts on link patterns, so the qKZ equation becomes a relationship between the components ψ_α. Can lead to complete calculation of ground state eigenvector.

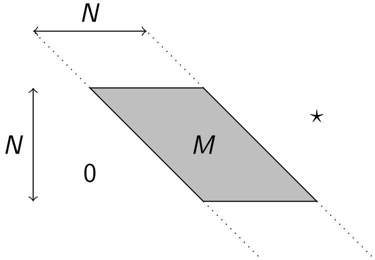
History

- [de Gier & Nienhuis, 2005]
  Periodic boundaries: Conjectured agreement between homogeneous ground state and degrees of the 'upper-upper' algebraic variety from [Knutson, 2003].
- [Di Francesco & Zinn-Justin, 2006]
  Refinement of conjecture to multidegrees, calculation of ground state
- [Knutson & Zinn-Justin, 2007]
  Proof of conjecture
- [Di Francesco, 2005]
  Closed boundaries: Calculation of ground state
- This talk
  Identified, closed, open, mixed boundaries: Agreement between ground state and multidegrees of variety with symmetries imposed.

Brauer loop scheme

The Brauer loop scheme (periodic)

Infinite upper-triangular (N, N)-periodic strip matrices:



$$\mathcal{M}_N^{\mathrm{p}} = \left\{ M \in U_N \mid MS^N = S^N M \right\} / \left\langle S^N \right\rangle$$

The Brauer loop scheme (periodic)

Brauer loop scheme:

$$E_N^{\mathrm{p}} = \left\{ M \in \mathcal{M}_N^{\mathrm{p}} \;\middle|\; (M^2)_{ij} = 0 \text{ for } i \leq j < i + N \right\}$$

The Brauer loop scheme (periodic)

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$$E_N^{\mathrm{p}} = \left\{ M \in \mathcal{M}_N^{\mathrm{p}} \;\middle|\; (M^2)_{ij} = 0 \text{ for } i \leq j < i + N \right\}$$

Theorem [Knutson–Zinn-Justin, 2007]

Brauer loop scheme is union of irreducible components (varieties), indexed by link patterns π:

$$E_N^{\mathrm{p}} = \bigcup_{\pi \in \mathrm{LP}_N^{\mathrm{p}}} E_\pi^{\mathrm{p}},$$

where

$$E_\pi^{\mathrm{p}} = \overline{\{M \in E_N^{\mathrm{p}} \mid (M^2)_{i,i+N} = (M^2)_{j,j+N} \Leftrightarrow i \in \{j, \pi(j)\} \;_{(1 \leq i,j \leq N)}\}}$$

Multidegrees (periodic)

To any variety X, associate a polynomial $\mathrm{mdeg}(X)$ in a canonical way.

Example

For a chain of hyperplanes so that $X = H_0 \subset \cdots \subset H_N = \mathcal{M}_N^{\mathrm{p}}$,

$$\mathrm{mdeg}(X) = \prod_{i=1}^{N} \mathrm{wt}_T(H_i/H_{i-1}).$$

Theorem [K–ZJ, 2007]

$$\mathrm{mdeg}(E_\pi^{\mathrm{p}}) \propto \psi_\pi^{\mathrm{p}},$$

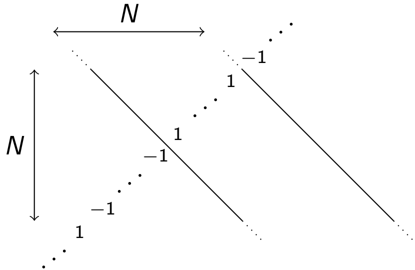
and therefore

$$\mathrm{mdeg}(E_N^{\mathrm{p}}) = \sum_{\pi \in \mathrm{LP}_N^{\mathrm{p}}} \mathrm{mdeg}(E_\pi^{\mathrm{p}}) \propto Z_N^{\mathrm{p}}.$$

The Brauer loop scheme (identified and closed)

$N = 2L$.
Symplectic form:

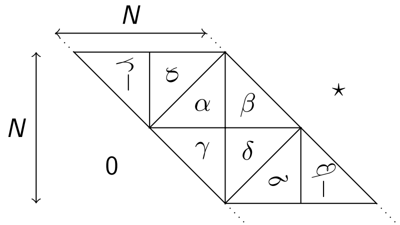
$J =$



The Brauer loop scheme (identified and closed)

$$E^{\mathrm{i}}_N = E^{\mathrm{p}}_N \cap \left\{ M \in \mathcal{M}^{\mathrm{p}}_N \;\middle|\; M = JM^T J \right\}$$
$$E^{\mathrm{c}}_N = E^{\mathrm{p}}_N \cap \left\{ M \in \mathcal{M}^{\mathrm{p}}_N \;\middle|\; M = -JM^T J \right\}$$

$E^{\mathrm{i}}_N \ni M =$



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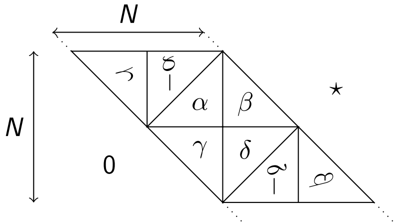
$E^{\mathrm{c}}_N \ni M =$ 

The Brauer loop scheme (identified and closed)

Our claim

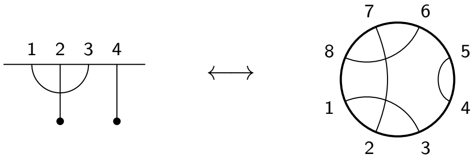
Irreducible components are indexed by link patterns π:

$$E_N^{\mathrm{i,c}} = \bigcup_{\pi \in \mathrm{LP}_{N/2}^{\mathrm{i,c}}} E_\pi^{\mathrm{i,c}},$$

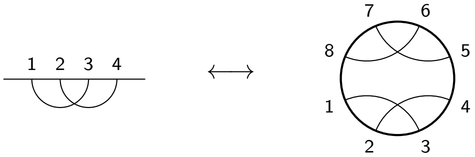
where

$$E_\pi^{\mathrm{i,c}} = \overline{\left\{ M \in E_N^{\mathrm{i,c}} \mid (M^2)_{i,i+N} = (M^2)_{j,j+N} \Leftrightarrow i \in \{j, \pi(j)\} \ {\scriptstyle (1 \le i,j \le N/2)} \right\}}$$

The Brauer loop scheme (identified and closed)



$$E_{\pi^{\mathsf{i}}} = E_{\pi^{\mathsf{p}}} \cap \left\{ M \in \mathcal{M}_N^{\mathsf{p}} \;\middle|\; M = JM^T J \right\}$$



$$E_{\pi^{\mathsf{c}}} = E_{\pi^{\mathsf{p}}} \cap \left\{ M \in \mathcal{M}_N^{\mathsf{p}} \;\middle|\; M = -JM^T J \right\}$$

Multidegrees (identified and closed)

Our claim

$$\mathrm{mdeg}(E_\pi^{\mathrm{i,c}}) \propto \psi_\pi^{\mathrm{i,c}},$$

and therefore

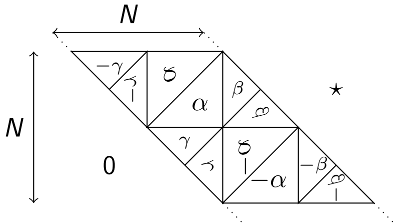
$$\mathrm{mdeg}(E_N^{\mathrm{i,c}}) = \sum_{\pi \in \mathrm{LP}_{N/2}^{\mathrm{i,c}}} \mathrm{mdeg}(E_\pi^{\mathrm{i,c}})$$
$$\propto Z_{N/2}^{\mathrm{i,c}}.$$

The Brauer loop scheme (open and mixed)

$N = 4L$.
Second Symplectic form:

$$J' = S^{N/4} J S^{-N/4}.$$

$$E_N^{\mathrm{o}} = E_N^{\mathrm{p}} \cap \left\{ M \in \mathcal{M}_N^{\mathrm{p}} \;\middle|\; M = JM^T J = J' M^T J' \right\}$$
$$E_N^{\mathrm{m}} = E_N^{\mathrm{p}} \cap \left\{ M \in \mathcal{M}_N^{\mathrm{p}} \;\middle|\; M = -JM^T J = J' M^T J' \right\}$$

$E_N^{\mathrm{o}} \ni M =$ 

The Brauer loop scheme (open and mixed)

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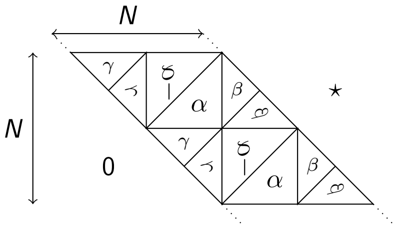
$$E_N^{\mathrm{m}} = E_N^{\mathrm{p}} \cap \left\{ M \in \mathcal{M}_N^{\mathrm{p}} \;\middle|\; M = -JM^T J = J' M^T J' \right\}$$

$E_N^{\mathrm{m}} \ni M =$ 

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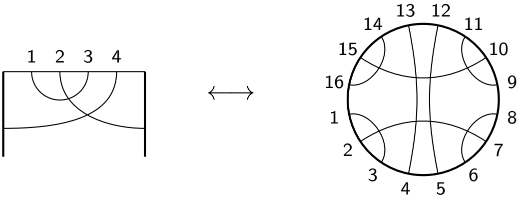
Irreducible components are indexed by link patterns π:

$$E_N^{\mathrm{o,m}} = \bigcup_{\pi \in \mathrm{LP}_{N/4}^{\mathrm{o,m}}} E_\pi^{\mathrm{o,m}},$$

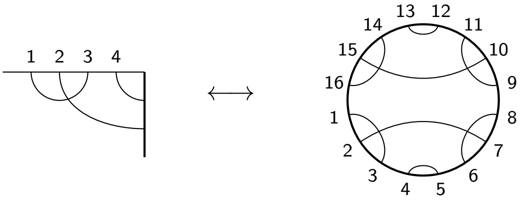
where

$$E_\pi^{\mathrm{o,m}} = \overline{\{M \in E_N^{\mathrm{o,m}} \mid (M^2)_{i,i+N} = (M^2)_{j,j+N} \Leftrightarrow i \in \{j, \pi(j)\} \ (1 \le i,j \le N/4)\}}$$

The Brauer loop scheme (open and mixed)



$$E_{\pi^{\mathrm{o}}} \;=\; E_{\pi^{\mathrm{p}}} \cap \left\{ M \in \mathcal{M}_N^{\mathrm{p}} \;\middle|\; M = JM^T J = J' M^T J' \right\}$$



$$E_{\pi^{\mathrm{m}}} \;=\; E_{\pi^{\mathrm{p}}} \cap \left\{ M \in \mathcal{M}_N^{\mathrm{p}} \;\middle|\; M = -JM^T J = J' M^T J' \right\}$$

Multidegrees (open and mixed)

Our claim

$$\text{mdeg}(E_\pi^{\text{o,m}}) \propto \psi_\pi^{\text{o,m}},$$

and therefore

$$\begin{aligned}\text{mdeg}(E_N^{\text{o,m}}) &= \sum_{\pi \in \text{LP}_{N/4}^{\text{o,m}}} \text{mdeg}(E_\pi^{\text{o,m}}) \\ &\propto Z_{N/4}^{\text{o,m}}.\end{aligned}$$

Conclusion

Brauer loop model with BCs:

- Link pattern π
- Ground state component ψ_π

Brauer matrix scheme with symmetries:

- Irreducible component indexed by π
- Multidegree of component m_π

Thank you for your attention