Finite size lattice results for the two-boundary Temperley–Lieb loop model

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Ph.D. Completion Seminar

14 June 2011

Project supervised by Jan de Gier

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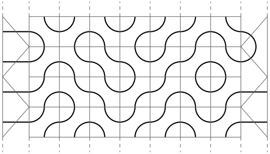
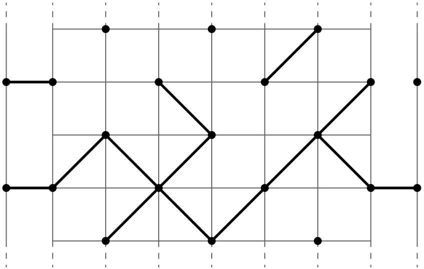
Outline

1. Introduction
   - The 2-boundary Temperley–Lieb algebra
   - The Temperley–Lieb loop model
2. The groundstate eigenvector $|\Psi\rangle$ and normalisation Z_L
   - Recursion relations
   - Result
3. Other work
   - The boundary-to-boundary correlation function F
   - A brief word on asymptotics
4. Conclusion and Outlook



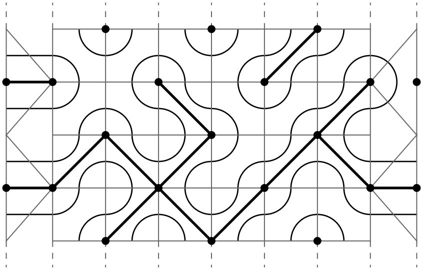
Motivation

Percolation model



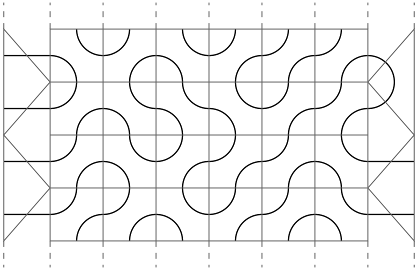
Motivation

Percolation model



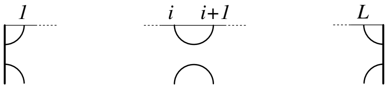
Motivation

Percolation model



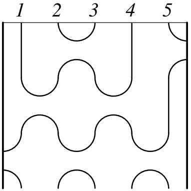
The 2-boundary Temperley–Lieb algebra

The generators:



e_0 e_i e_L

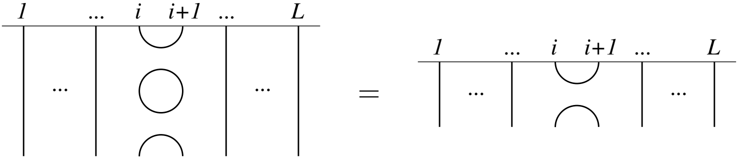
Concatenation:



$e_2 e_5 e_1 e_3 e_0 e_2 e_4$

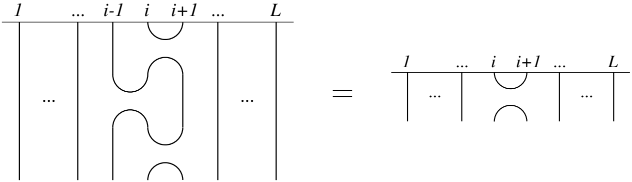
The 2-boundary Temperley–Lieb algebra: Rules

⋆ Closed loops are removed



$$e_i^2 = e_i$$

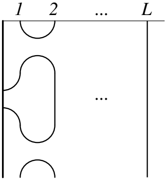
⋆ Strings are pulled tight



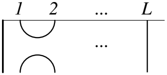
$$e_i e_{i\pm1} e_i = e_i$$

The 2-boundary Temperley–Lieb algebra: Rules

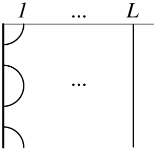
⋆ Loops connected to one boundary are removed



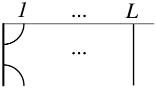
$=$



$$e_1 e_0 e_1 = e_0$$



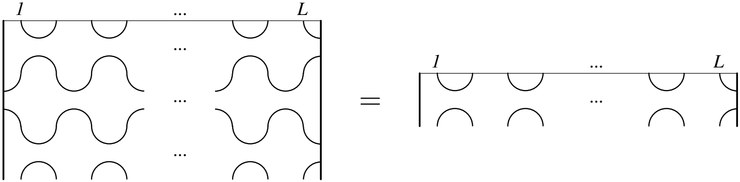
$=$



$$e_0^2 = e_0$$

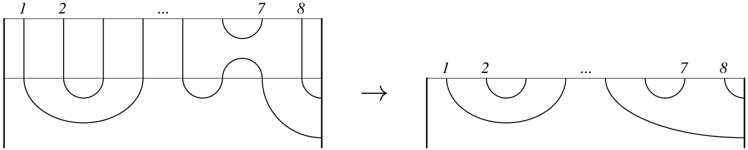
The 2-boundary Temperley–Lieb algebra: Rules

⋆ Pairs of loops connecting left boundary to right are removed

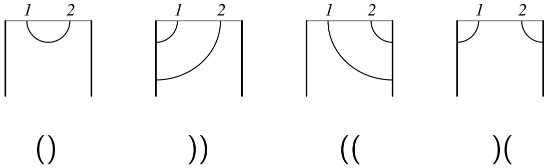


$$(e_1 e_3 \ldots)\,(e_0 e_2 \ldots)\,(e_1 e_3 \ldots) \;=\; (e_1 e_3 \ldots)$$
$$(e_0 e_2 \ldots)\,(e_1 e_3 \ldots)\,(e_0 e_2 \ldots) \;=\; (e_0 e_2 \ldots)$$

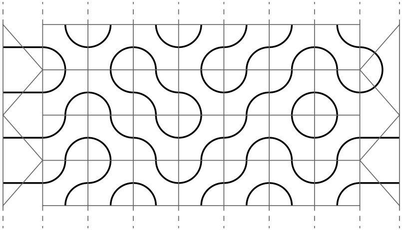
Link Patterns



$L = 2$:



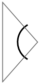
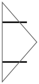
The Temperley–Lieb loop model



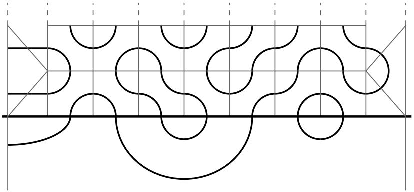
Bulk tiles:

 and 

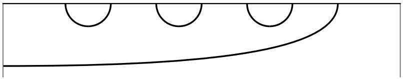
Boundary tiles (left):

 and 

The Temperley–Lieb loop model



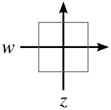
The Temperley–Lieb loop model



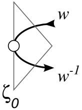
$$\alpha = ()()(())$$

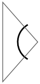
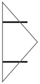
Relative probability of α is ψ_α

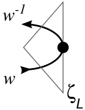
$$|\Psi\rangle = \sum_\alpha \psi_\alpha |\alpha\rangle$$

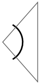
$R(z, w) =$ 

$= a(\frac{z}{w})$  $+\ b(\frac{z}{w})$ 

$K_0(w) =$ 

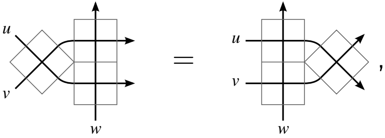
$= c(qw, \zeta_0)$  $+\ d(qw, \zeta_0)$ 

$K_L(w) =$ 

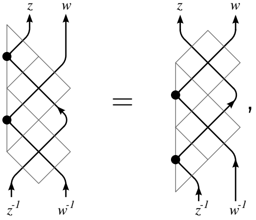
$= c(w, \zeta_L)$  $+\ d(w, \zeta_L)$ 

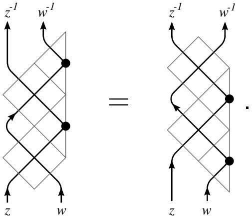
Baxterisation: Relations

Yang–Baxter equation:



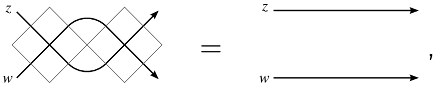
Reflection equation:



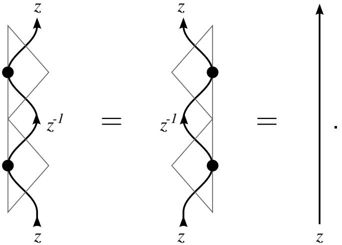


Unitarity:

$R(z,w)R(w,z) = 1$

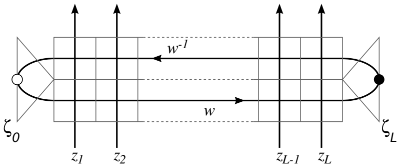


$K(w)K(\frac{1}{w}) = 1$



Transfer Matrix

$$T_L(w; \mathbf{z}; \zeta_0, \zeta_L) =$$



$$T_L|\Psi\rangle = |\Psi\rangle$$

$$[T_L(w), T_L(u)] = 0$$

$$|\Psi(\mathbf{z}; \zeta_0, \zeta_L)\rangle = \sum_{\alpha} \psi_\alpha(\mathbf{z}; \zeta_0, \zeta_L)\ |\alpha\rangle$$

Eigenvector and Normalisation

$$|\Psi(\mathbf{z};\zeta_0,\zeta_L)\rangle = \sum_\alpha \psi_\alpha(\mathbf{z};\zeta_0,\zeta_L)\ |\alpha\rangle$$

ψ_α are polynomials

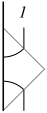
$$Z_L = \sum_\alpha \psi_\alpha$$

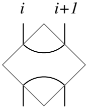
ψ_α / Z_L is *absolute* probability of link pattern α

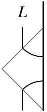
Method of proof:

- Recursions from system size L to $L - 1$ and $L - 2$
- Symmetries
- Degree
- Small system sizes

Alternate Baxterisation

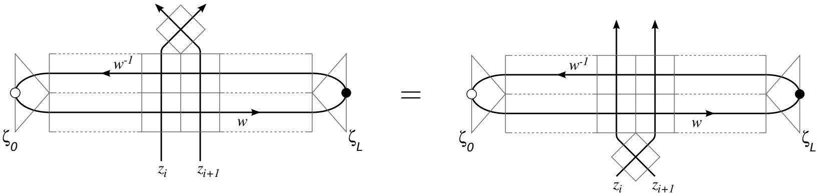
$e_0 =$ 

$e_i =$ 

$e_L =$ 

$$\check{R}_i(z) = a(z) + b(z)e_i, \qquad 1 \le i \le L-1$$
$$\check{K}_k(z) = c(z,\zeta_k) + d(z,\zeta_k)e_k, \qquad k = 0, L$$

Interlacing Conditions



$$\check{R}_i\left(\frac{z_i}{z_{i+1}}\right) T_L(\ldots, z_i, z_{i+1}, \ldots) = T_L(\ldots, z_{i+1}, z_i, \ldots)\check{R}_i\left(\frac{z_i}{z_{i+1}}\right)$$

$$\check{K}_0\left(\frac{1}{z_1}\right) T_L(z_1, \ldots) = T_L\left(\frac{1}{z_1}, \ldots\right)\check{K}_0\left(\frac{1}{z_1}\right)$$

$$\check{K}_L(z_L)\, T_L(\ldots, z_L) = T_L\left(\ldots, \frac{1}{z_L}\right)\check{K}_L(z_L)$$

The q-deformed Knizhnik–Zamolodchikov equation

$$\check{R}_i\left(\frac{z_i}{z_{i+1}}\right)|\Psi\rangle = \pi_i|\Psi\rangle \qquad 1 \leq i \leq L-1$$

$$\check{K}_0\left(\frac{1}{z_1}, \zeta_0\right)|\Psi\rangle = \pi_0|\Psi\rangle$$

$$\check{K}_L(z_L, \zeta_L)|\Psi\rangle = \pi_L|\Psi\rangle$$

$$\pi_i \; f(z_i, z_{i+1}) = f(z_{i+1}, z_i) \qquad 1 \leq i \leq L-1$$

$$\pi_0 \; f(z_1) = f\left(\frac{1}{z_1}\right)$$

$$\pi_L \; f(z_L) = f\left(\frac{1}{z_L}\right)$$

The q-deformed Knizhnik–Zamolodchikov equation

$$\check{R}_i\left(\frac{z_i}{z_{i+1}}\right)|\Psi\rangle = \pi_i|\Psi\rangle$$

$$\left[a\left(\frac{z_i}{z_{i+1}}\right) + b\left(\frac{z_i}{z_{i+1}}\right)e_i\right]|\Psi\rangle = \pi_i|\Psi\rangle$$

The q-deformed Knizhnik–Zamolodchikov equation

$$\check{R}_i\left(\tfrac{z_i}{z_{i+1}}\right)|\Psi\rangle = \pi_i|\Psi\rangle$$

$$\left[a\left(\tfrac{z_i}{z_{i+1}}\right) + b\left(\tfrac{z_i}{z_{i+1}}\right)e_i\right]|\Psi\rangle = \pi_i|\Psi\rangle$$

$$\Rightarrow \qquad e_i|\Psi\rangle = a_i|\Psi\rangle$$

$$a_i = \frac{1}{b\left(\tfrac{z_i}{z_{i+1}}\right)}\left(\pi_i - a\left(\tfrac{z_i}{z_{i+1}}\right)\right)$$

The q-deformed Knizhnik–Zamolodchikov equation

$$\check{K}_L(z_L, \zeta_L)|\Psi\rangle = \pi_L|\Psi\rangle$$
$$[c(z_L, \zeta_L) + d(z_L, \zeta_L)e_L]\,|\Psi\rangle = \pi_L|\Psi\rangle$$

The q-deformed Knizhnik–Zamolodchikov equation

$$\check{K}_L(z_L, \zeta_L)|\Psi\rangle = \pi_L|\Psi\rangle$$

$$[c(z_L, \zeta_L) + d(z_L, \zeta_L)e_L]\,|\Psi\rangle = \pi_L|\Psi\rangle$$

$$\Rightarrow \qquad e_L|\Psi\rangle = a_L|\Psi\rangle$$

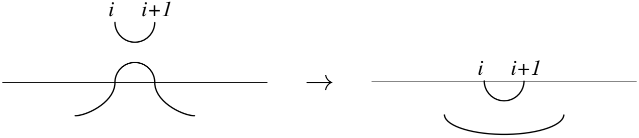
$$a_L = \frac{1}{d(z_L, \zeta_L)}\left(\pi_L - c(z_L, \zeta_L)\right)$$

The q-deformed Knizhnik–Zamolodchikov equation

$$\sum_{\alpha} \psi_{\alpha} \Big(e_i |\alpha\rangle \Big) = \sum_{\alpha} \Big(a_i \psi_{\alpha} \Big) |\alpha\rangle, \quad \forall i$$

The q-deformed Knizhnik–Zamolodchikov equation

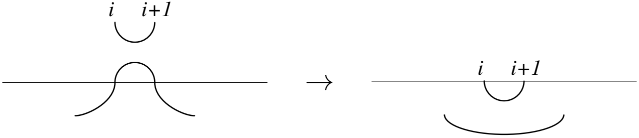
$$\sum_{\alpha} \psi_\alpha \Big(e_i |\alpha\rangle \Big) = \sum_{\alpha} \Big(a_i \psi_\alpha \Big) |\alpha\rangle, \quad \forall i$$



$$a_i \psi_\alpha = 0 \qquad \text{if } \alpha \neq \ldots () \ldots$$

The q-deformed Knizhnik–Zamolodchikov equation

$$\sum_\alpha \psi_\alpha \Big(e_i |\alpha\rangle \Big) = \sum_\alpha \Big(a_i \psi_\alpha \Big) |\alpha\rangle, \quad \forall i$$



$$a_i \psi_\alpha = 0 \qquad \text{if } \alpha \neq \ldots () \ldots$$

$$\Rightarrow \quad \psi_\alpha = \left(\frac{q z_i}{z_{i+1}} - \frac{z_{i+1}}{q z_i} \right) S(z_i, z_{i+1})$$

$$q^3 = 1$$

Recursions

If $\alpha \neq \ldots () \ldots$

$$\psi_\alpha = \left(\frac{qz_i}{z_{i+1}} - \frac{z_{i+1}}{qz_i} \right) S(z_i, z_{i+1})$$

$$\Rightarrow \psi_\alpha(z_{i+1} = qz_i) = 0$$

Recursions

If $\alpha \neq \ldots () \ldots$

$$\psi_\alpha = \left(\frac{qz_i}{z_{i+1}} - \frac{z_{i+1}}{qz_i} \right) S(z_i, z_{i+1})$$

$$\Rightarrow \psi_\alpha(z_{i+1} = qz_i) = 0$$

At $z_{i+1} = qz_i$, only non-zero ψ_α have

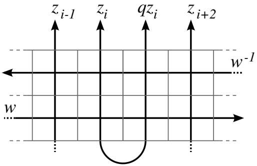
$$\alpha = \varphi_i \beta$$

E.g.

$$\varphi_3 \; ()((= ()(()($$

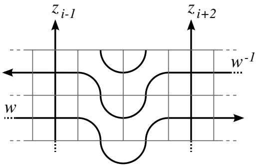
Recursions

$$T_L|\Psi\rangle_L\Big|_{z_{i+1}=qz_i} =$$

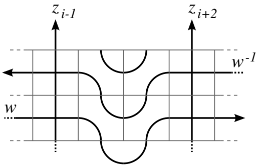


Recursions

$$\color{darkred}{T_L}\color{black}{|}\color{darkblue}{\Psi}\color{black}{\rangle_L\Big|_{z_{i+1}=qz_i}} =$$



$$T_L |\Psi\rangle_L \Big|_{z_{i+1}=qz_i} =$$



$$T_L(z_{i+1} = qz_i) \circ \varphi_i = \varphi_i \circ T_{L-2}(\hat{z}_i, \hat{z}_{i+1})$$

$$|\Psi(z_{i+1} = qz_i)\rangle_L = p(z_i)\ \varphi_i |\Psi(\hat{z}_i, \hat{z}_{i+1})\rangle_{L-2}$$

Recursions

Where $\alpha = \varphi_i \beta$

$$\psi_\alpha(z_{i+1} = qz_i) = p(z_i)\ \psi_\beta(\hat{z}_i, \hat{z}_{i+1})$$

Thus

$$Z_L(z_{i+1} = qz_i) = p(z_i)\ Z_{L-2}(\hat{z}_i, \hat{z}_{i+1})$$

Recursions: Boundary

If $\alpha \neq \,)\ldots$

$$a_0 \psi_\alpha = 0$$

Recursions: Boundary

If $\alpha \neq$)...

$$a_0 \psi_\alpha = 0$$

$$\begin{aligned} \psi_\alpha &= \left(\frac{q\zeta_0}{z_1} - \frac{z_1}{q\zeta_0}\right)\left(\frac{q}{z_1\zeta_0} - \frac{z_1\zeta_0}{q}\right)\widetilde{S}(z_1) \\ &= k(z_1, \zeta_0)\,\widetilde{S}(z_1) \end{aligned}$$

Recursions: Boundary

If $\alpha \neq$) . . .

$$a_0 \psi_\alpha = 0$$

$$\begin{aligned}\psi_\alpha &= \left(\frac{q\zeta_0}{z_1} - \frac{z_1}{q\zeta_0}\right)\left(\frac{q}{z_1\zeta_0} - \frac{z_1\zeta_0}{q}\right)\widetilde{S}(z_1)\\ &= k(z_1, \zeta_0)\,\widetilde{S}(z_1)\end{aligned}$$

$$\Rightarrow \quad \psi_\alpha(z_1 = q\zeta_0) = 0$$

Recursions: Boundary

$$\widetilde{\varphi}_0 \ )((=))(($$

.

$$T_L(z_1 = q\zeta_0) \circ \widetilde{\varphi}_0 = \widetilde{\varphi}_0 \circ T_{L-1}(\hat{z}_1)$$

$$|\Psi(z_1 = q\zeta_0)\rangle_L = r_0(\zeta_0)\ \widetilde{\varphi}_0\ |\Psi(\hat{z}_1)\rangle_{L-1}$$

$$Z_L(z_1 = q\zeta_0) = r_0(\zeta_0)\ Z_{L-1}(\hat{z}_1)$$

Result: Normalisation

$$Z_L = \tau_L(\mathbf{z})\ \tau_{L+1}(\zeta_0, \mathbf{z})\ \tau_{L+1}(\mathbf{z}, \zeta_0)\ \tau_{L+2}(\zeta_0, \mathbf{z}, \zeta_L)$$

$$\tau_L(z_1, \ldots, z_L) = \frac{\det\left[z_i^{2\mu_j} - z_i^{-2\mu_j}\right]}{\det\left[z_i^{2\delta_j} - z_i^{-2\delta_j}\right]}$$

$$\mu = (\ldots, 7, 5, 4, 2, 1)$$
$$\delta = (\ldots, 5, 4, 3, 2, 1)$$

Components

$a_1\psi_{(\cdots(} = 0$:

$$\psi_{(\cdots(} = \left(\frac{qz_1}{z_2} - \frac{z_2}{qz_1}\right)\ldots$$

Components

$a_0 \psi_{(\cdots(} = 0$:

$$\psi_{(\cdots(} = \left(\frac{qz_1}{z_2} - \frac{z_2}{qz_1}\right) k(z_1, \zeta_0) \ldots$$

Components

$a_0\psi_{(\cdots(} = a_1\psi_{(\cdots(} = 0$:

$$\psi_{(\cdots(} = \left(\frac{qz_1}{z_2} - \frac{z_2}{qz_1}\right) k(z_1, \zeta_0)k(z_2, \zeta_0)\ldots$$

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$$= k(z_2,z_1)k(z_1,\zeta_0)k(z_2,\zeta_0) \ldots$$

Components

$a_0\psi_{(\cdots(} = \ldots = a_{L-1}\psi_{(\cdots(} = 0$:

$$\psi_{(\cdots(} = \prod_{i=1}^{L} k(z_i, \zeta_0) \prod_{1\leq i<j\leq L} k(z_j, z_i)\ f(z_1, \ldots, z_L)$$

Components

$a_0\psi_{(\cdots(} = \ldots = a_{L-1}\psi_{(\cdots(} = 0$:

$$\psi_{(\cdots(} = \prod_{i=1}^{L} k(z_i,\zeta_0) \prod_{1\le i<j\le L} k(z_j,z_i)\; f(z_1,\ldots,z_L)$$

$a_1\psi_{)\cdots)} = \ldots = a_L\psi_{)\cdots)} = 0$:

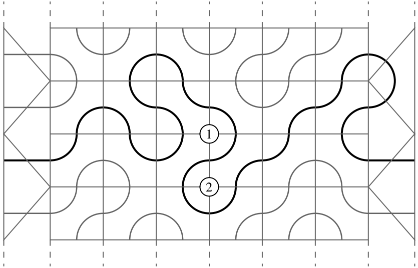
$$\psi_{)\cdots)} = \prod_{i=1}^{L} k(1/z_i,\zeta_L) \prod_{1\le i<j\le L} k(1/z_i,z_j)\; \tilde{f}(z_1,\ldots,z_L)$$

Result: Components

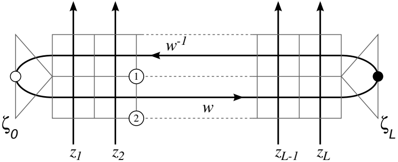
$$f(z_1, \ldots, z_L) = \tau_L(\mathbf{z})\tau_{L+1}(\mathbf{z}, \zeta_L)$$

$$\tilde{f}(z_1, \ldots, z_L) = \tau_L(\mathbf{z})\tau_{L+1}(\zeta_0, \mathbf{z})$$

The boundary-to-boundary correlation function



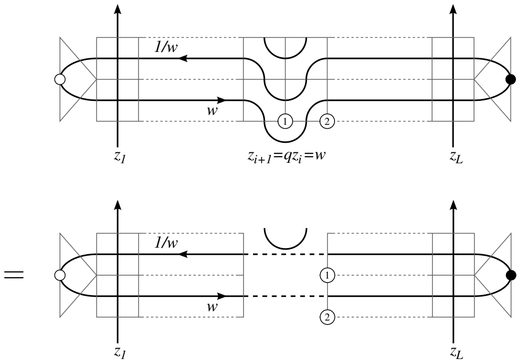
The boundary-to-boundary correlation function

$\widehat{F}_L =$ 

Expectation value

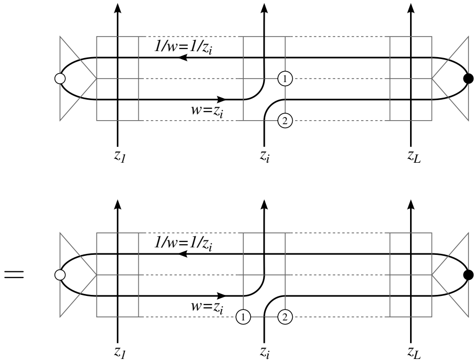
$$F_L = \frac{\langle \Psi | \widehat{F}_L | \Psi \rangle}{Z_L^2}$$

Extra relations



$$F_L^{(\text{hor.})}(z_{i+1} = w; z_i = w/q) = F_{L-2}^{(\text{vert.})}(\hat{z}_i, \hat{z}_{i+1})$$

Extra relations



$$F_L^{\text{(vert.)}}(w = z_i) = F_L^{\text{(hor.)}}$$

Result

$$F_L = \frac{\partial}{\partial z_i} \log \left[\frac{\tau_{L+1}(\zeta_0, \mathbf{z})\ \tau_{L+1}(\mathbf{z}, \zeta_0)}{\tau_{L+2}(\zeta_0, \mathbf{z}, \zeta_L)\ \tau_L(\mathbf{z})} \right]$$

$$\tau_L(z_1, \ldots, z_L) = \frac{\det \left[z_i^{2\mu_j} - z_i^{-2\mu_j} \right]}{\det \left[z_i^{2\delta_j} - z_i^{-2\delta_j} \right]}$$

Asymptotics

Homogeneous limit $L \to \infty$:

$$\lim_{L \to \infty} \tau_L(\zeta_0, \zeta_L, 1, \ldots, 1)$$

Separation of Variables: Sklyanin's scheme

Separating operator $\mathcal{S}$

$$\mathcal{S}_k \; \tau_L(z_1, \ldots, z_k, 1, \ldots, 1) = \prod_{i=1}^{k} q(z_i) \; \tau_L(1, \ldots, 1).$$

Asymptotics

Homogeneous limit $L \to \infty$:

$$\lim_{L\to\infty} \tau_L(\zeta_0, \zeta_L, 1, \ldots, 1) = \lim_{L\to\infty} \mathcal{S}_2^{-1}\left[q(\zeta_0)q(\zeta_L)\tau_L(1, \ldots, 1)\right]$$

$$\left[\prod_{n=1}^{L}(Z^2 - \mu_n^2)\right] q(z) = 0$$

where

$$Z = \frac{\partial}{\partial z} + \frac{(z+1)^2 L - 2z}{z^2 - 1}$$

Conclusion

Finite size 2 boundary Temperley–Lieb loop model calculations:

- Eigenvector $|\Psi\rangle$ (some components)
- Normalisation Z_L
- Boundary-to-boundary correlation function F
- Separating operator (asymptotics)

Outlook:

- Razumov–Stroganov type conjectures
- Asymptotics

arXiv:0901.2961 arXiv:1004.4037 arXiv:1009.2831

Thank you