The two-boundary Brauer loop model

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Université de Genève

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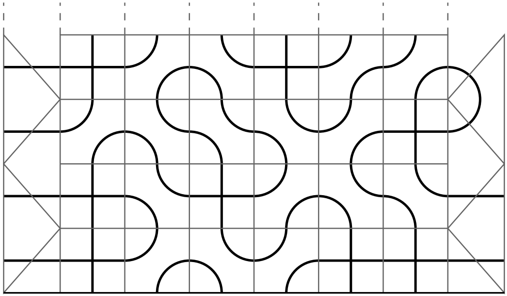
Les Diablerets

Work-in-progress in collaboration with Paul Zinn-Justin

Introduction

Introduction: The Brauer model

Loop model on a semi-infinite lattice:



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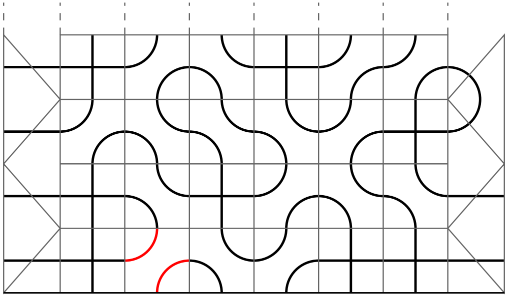

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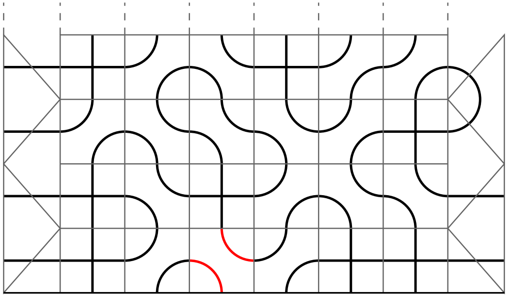

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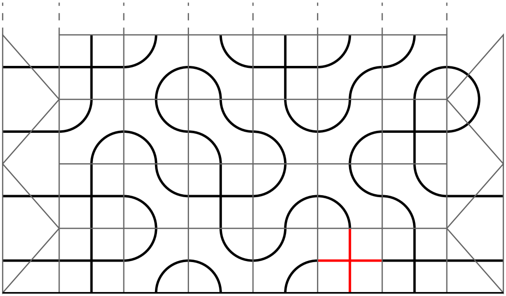

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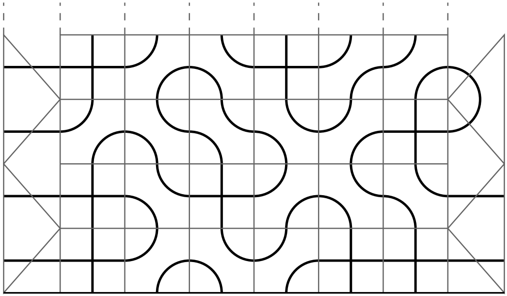
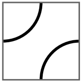
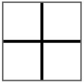


$R = a$  $+ b$  $+ c$ 

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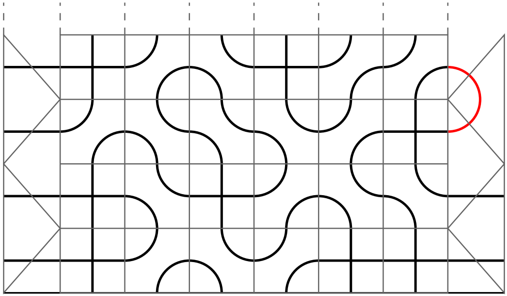

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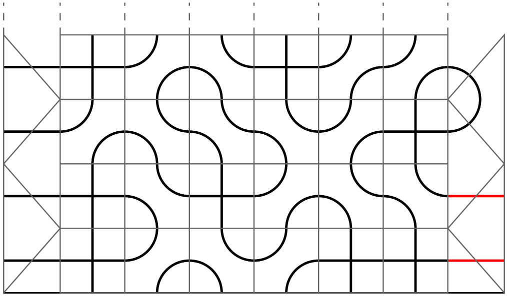

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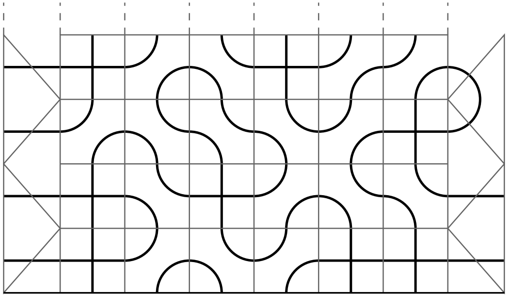
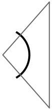
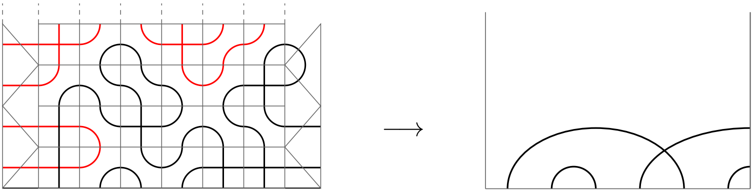


$$K = k_1$$  $$+ k_2$$ 

Link patterns

Sites can be connected to each other or to the boundary.
Ignore closed loops and loops connected only to the boundaries:



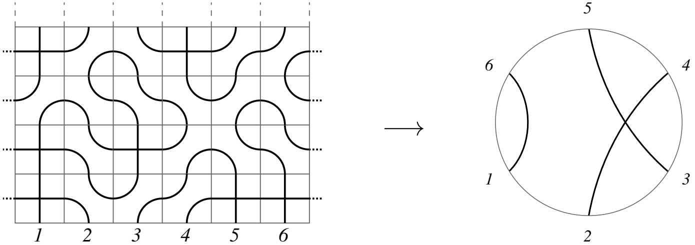
"Probability" vector, with α a link pattern:

$$|\Psi\rangle = \sum_{\alpha} \psi_\alpha |\alpha\rangle$$

Sum rule:

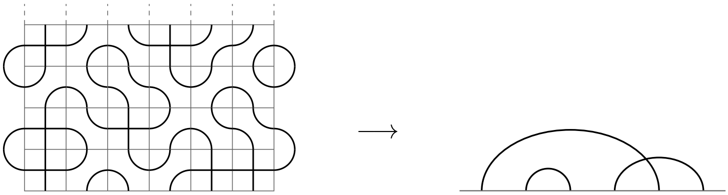
$$Z_L = \sum_{\alpha} \psi_\alpha$$

Other boundary conditions: Periodic



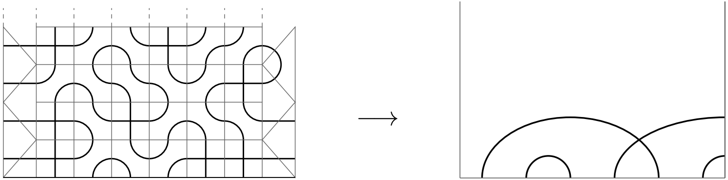
- [de Gier & Nienhuis, 2005]: Conjectured agreement between probabilities of link patterns and degrees of the 'upper-upper' algebraic variety from [Knutson, 2003].
- [Di Francesco & Zinn-Justin, 2006]: Refinement of conjecture, calculations of ψ_α, Z_L and sum of permutation-type components.
- [Knutson & Zinn-Justin, 2007]: Proof of conjecture, involving permutation-type components.

Other boundary conditions: Reflecting



- [Di Francesco, 2005]: Calculations of ψ_α, Z_L and sum of permutation-type components.

This talk: Open boundaries

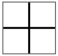


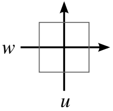
- Calculations of (some) ψ_α, Z_L and sum of permutation-type components. (Almost complete proof)

Details

R Matrix

Introduce inhomogeneities.

$R(w-u) = a(w-u)$  $+\, b(w-u)$  $+\, c(w-u)$ 

$=$ 

Probability of configurations on a face:

$$a(z) = \frac{2(z-1)}{(z+1)(z-2)}, \quad b(z) = \frac{-2z}{(z+1)(z-2)}, \quad c(z) = \frac{z(z-1)}{(z+1)(z-2)}$$

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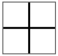
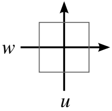
$R(w-u) = a(w-u)$  $+\, b(w-u)$  $+\, c(w-u)$ 

$=$ 

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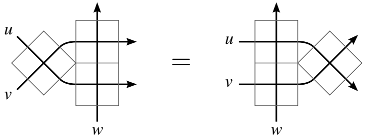
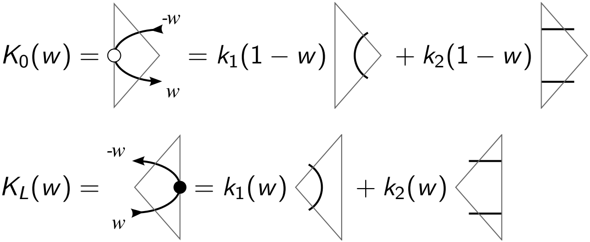
Chosen so that Yang–Baxter equation holds:





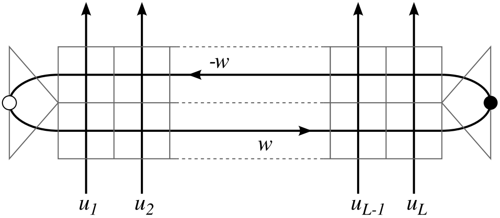
Inhomogeneous probabilities:

$$k_1(w) = \frac{1 - 2w}{1 + 2w}, \quad k_2(w) = \frac{4w}{1 + 2w}$$

Chosen so boundary YBE holds.

Transfer Matrix

Probabilities of configurations on two rows of the lattice:

$T(w; u_1, \ldots, u_L) =$ 

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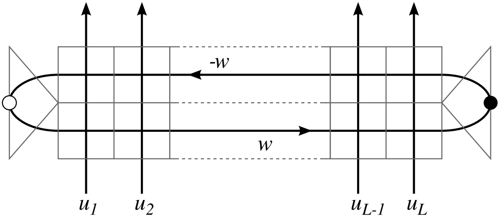
$T(w; u_1, \ldots, u_L) =$ 

Ground state:

$$|\Psi(u_1, \ldots, u_L)\rangle = \sum_\alpha \psi_\alpha(u_1, \ldots, u_L)|\alpha\rangle$$

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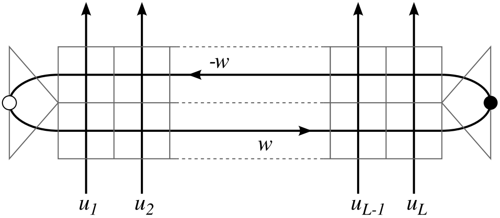
$T(w; u_1, \ldots, u_L) =$ 

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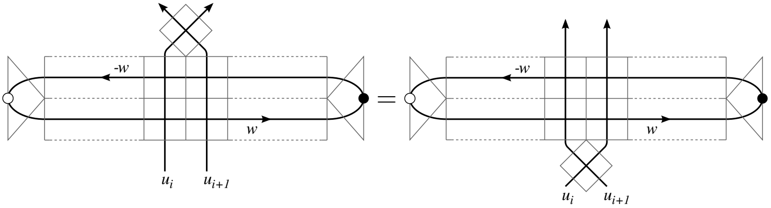
Eigenvalue of 1:

$$T(w; u_1, \ldots, u_L)\ |\Psi(u_1, \ldots, u_L)\rangle = |\Psi(u_1, \ldots, u_L)\rangle$$

Interlacing relation (bulk)

YBE implies:



Equivalently,

$$R(u_i - u_{i+1})T(w, \dots, u_i, u_{i+1}, \dots) = T(w, \dots, u_{i+1}, u_i, \dots)R(u_i - u_{i+1})$$

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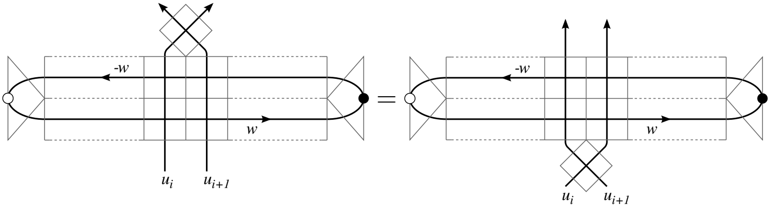

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Leading to *q*-deformed Knizhnik–Zamolodchikov equation:

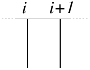
$$R(u_i - u_{i+1})|\Psi(\dots, u_i, u_{i+1}, \dots)\rangle = |\Psi(\dots, u_{i+1}, u_i, \dots)\rangle$$

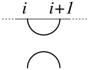
qKZ equation (bulk)

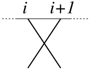
$$R(u_i - u_{i+1})|\Psi(\dots, u_i, u_{i+1}, \dots)\rangle = |\Psi(\dots, u_{i+1}, u_i, \dots)\rangle$$

R in terms of Brauer algebra generators:

$R(u_i - u_{i+1})$

$= a(u_i - u_{i+1})$ 

$+ b(u_i - u_{i+1})$ 

$+ c(u_i - u_{i+1})$ 

$= a_i + b_i\, e_i + c_i\, f_i$

e_i & f_i act on $|\alpha\rangle$, so qKZ equation becomes a relationship between the components of $|\Psi\rangle$.

qKZ equation (bulk)

$$(a_i + b_i e_i + c_i f_i)\ |\Psi\rangle = \pi_i\ |\Psi\rangle \qquad \big(= |\Psi(u_{i+1}, u_i)\rangle\big)$$

Three cases for $|\alpha\rangle$:

- $f_i|\alpha\rangle = |\alpha\rangle \quad \& \quad \alpha$ has no loop from i to $i+1$,

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- $f_i|\alpha\rangle = |\beta\rangle \neq |\alpha\rangle$, $\Rightarrow$ α has no loop from i to $i+1$,

$$a_i\psi_\alpha + c_i\psi_\beta = \pi_i\psi_\alpha$$
$$a_i\psi_\beta + c_i\psi_\alpha = \pi_i\psi_\beta$$

- α has a loop from i to $i+1$, $\Rightarrow$ $f_i|\alpha\rangle = |\alpha\rangle$,

$$(a_i + c_i)\psi_\alpha + b_i \sum_{\gamma : e_i|\gamma\rangle = |\alpha\rangle} \psi_\gamma = \pi_i\psi_\alpha$$

qKZ equation (bulk)

First case: $f_i|\alpha\rangle = |\alpha\rangle$ & α has no loop from i to $i+1$,

$$(a_i + c_i)\psi_\alpha = \pi_i \psi_\alpha$$

More detail:

$$\begin{aligned}\psi_\alpha(u_{i+1}, u_i) &= (a_i + c_i)\psi_\alpha(u_i, u_{i+1}) \\ &= \frac{(u_{i+1} - u_i + 1)(u_{i+1} - u_i - 2)}{(u_i - u_{i+1} + 1)(u_i - u_{i+1} - 2)}\psi_\alpha(u_i, u_{i+1})\end{aligned}$$

$$\begin{aligned}\Rightarrow \quad \psi_\alpha(u_i, u_{i+1}) &= (u_i - u_{i+1} + 1)(u_i - u_{i+1} - 2) S^{\{i,i+1\}}(u_i, u_{i+1}) \\ &= r(u_i - u_{i+1}) S^{\{i,i+1\}}(u_i, u_{i+1})\end{aligned}$$

Results

Special Components

qKZ directly implies:

$$\psi_{\underbrace{L+1,\ldots,L+1}_{k},\underbrace{0,\ldots,0}_{L-k}} = \prod_{i=1}^{k}(1-2u_i)\prod_{i=k+1}^{L}(1+2u_i)$$

$$\times \prod_{1\le i<j\le k} r(u_i-u_j)r(-u_i-u_j) \prod_{k+1\le i<j\le L} r(u_i-u_j)r(u_i+u_j)$$

$$\times \prod_{i=1}^{k}\prod_{j=k+1}^{L}[(u_i-u_j+1)(-u_i-u_j+1)\ (u_i+u_j+1)(-u_i+u_j+1)]$$

$$\times\ S^{\{0,\ldots,k\},\{k+1,\ldots,L\}}(u_1,\ldots,u_L),$$

Conjecture (bound on degree): $S=1$.

Recursion (bulk)

Can also show (no details given here):

$$|\Psi_L(\dots, u_i, u_i+1, \dots)\rangle = p(u_i; \dots, \hat{u}_i, \hat{u}_{i+1}, \dots)|\Psi_{L-2}(\dots, \hat{u}_i, \hat{u}_{i+1}, \dots)\rangle$$

Implying

$$Z_L(\dots, u_i, u_i+1, \dots) = p(u_i; \dots, \hat{u}_i, \hat{u}_{i+1}, \dots) Z_{L-2}(\dots, \hat{u}_i, \hat{u}_{i+1}, \dots).$$

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Implying

$$Z_L(\dots,u_i,u_i+1,\dots) = p(u_i;\dots,\hat{u}_i,\hat{u}_{i+1},\dots)Z_{L-2}(\dots,\hat{u}_i,\hat{u}_{i+1},\dots).$$

With the conjectured bound on the degree, we have

$$p(u_i;\dots) = (1-2u_i)^2(1+2(u_i+1))^2 \prod_{j\neq i,i+1} r(u_j-u_i)^2 r(u_j+u_i+1)^2$$

This recursion is useful for proving the sum rule (next slide).

Sum Rule

With the conjectured bound on the degree,

$$Z_{2n} = \prod_{1 \le i < j \le 2n} b(u_i, u_j)^2 \ \det \left[\frac{(-5 + 2u_i^2 + u_j^2)}{b(u_i, u_j)} \right]_{1 \le i,j \le 2n}$$

$$Z_{2n-1} = 2 \prod_{1 \le i < j \le 2n-1} b(u_i, u_j)^2$$
$$\times \det \begin{bmatrix} \left[\frac{(-5+2u_i^2+2u_j^2)}{b(u_i,u_j)} \right]_{1 \le i,j \le 2n-1} & [-1]_{1 \le j \le 2n-1} \\ [1]_{1 \le i \le 2n-1} & 0 \end{bmatrix}$$

where

$$b(u_i, u_j) = \frac{(1 - (u_i - u_j)^2)(1 - (u_i + u_j)^2)}{u_i^2 - u_j^2}$$

Permutation Sector

$W_{2n} = \sum \psi_\alpha$ where α is a permutation between sites $1, \ldots, n$ and $n+1, \ldots, 2n$.

$$W_{2n}(u_1, \ldots, u_{2n}) = \sqrt{Z_{2n}} \prod_{i=1}^{n} (1 - 2u_i)(1 + 2u_{n+i}) \times \prod_{1 \le i < j \le n} r(u_i - u_j) r(-u_i - u_j) r(u_{n+i} - u_{n+j}) r(u_{n+i} + u_{n+j})$$

Conclusion

- Need proof of degree
- Still working on proper proof for W_{2n}
- Working on model with general closed loop weight, $n = 2$ corresponds to non-crossing case
- Looking for connection to algebraic varieties

Thank you