Separation of Variables for the Symplectic Character

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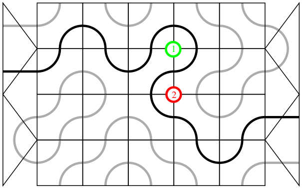
12 November, 2010

Work done in collaboration with Jan de Gier
Available at arXiv:1009.2831

Motivation

Lattice Model - O($n = 1$) loop model



Correlation function:

$$X_L^{(k)} = c_L \; z_k \frac{\partial}{\partial z_k} \log \left[\frac{\chi_\lambda(\zeta_1, z_1, \ldots, z_L) \; \chi_\lambda(z_1, \ldots, z_L, \zeta_2)}{\chi_\lambda(z_1, \ldots, z_L) \; \chi_\lambda(\zeta_1, z_1, \ldots, z_L, \zeta_2)} \right]$$

Partition function:

$$Z_L = \chi_\lambda(z_1, \ldots, z_L) \; \chi_\lambda(\zeta_1, z_1, \ldots, z_L) \\ \times \chi_\lambda(z_1, \ldots, z_L, \zeta_2) \; \chi_\lambda(\zeta_1, z_1, \ldots, z_L, \zeta_2)$$

Q-operator method (Kuznetsov, Mangazeev, Sklyanin 2003)

Polynomials P_λ are eigenfunctions of set of commuting Hamiltonians H_j,

$$H_j P_\lambda(x_1, \ldots, x_L) = h_j(\lambda) P_\lambda(x_1, \ldots, x_L)$$

Want a version where the polynomial is factorised, ie

$$\widetilde{H}_j \prod_{i=1}^{L} q_\lambda(x_i) = h_j(\lambda) \prod_{i=1}^{L} q_\lambda(x_i)$$

Want to find a 'Separating operator' $\mathcal{S}$,

$$\mathcal{S} P_\lambda(x_1, \ldots, x_L) = \prod_{i=1}^{L} q_\lambda(x_i) \; P_\lambda(1, \ldots, 1),$$

which allows transformation of spectral problem;

$$\left(\mathcal{S} H_j \mathcal{S}^{-1}\right) \mathcal{S} P_\lambda(x_1, \ldots, x_L) = h_j(\lambda) \; \mathcal{S} P_\lambda(x_1, \ldots, x_L)$$

Q-operator method

$$\mathcal{S} P_\lambda(x_1, \ldots, x_L) = \prod_{i=1}^{L} q_\lambda(x_i)\; P_\lambda(1, \ldots, 1)$$

Introduce Q-operator:

$$Q_z P_\lambda(x_1, \ldots, x_L) = q_\lambda(z) P_\lambda(x_1, \ldots, x_L)$$

Separating operator is a chain of Q-operators

$$\mathcal{S} = \left.\left(Q_{z_1} \ldots Q_{z_L}\right)\Big|_{x_1,\ldots,x_L=1}\right|_{z_j=x_j}$$

Q-operator method

$$\mathcal{S} P_\lambda(x_1, \ldots, x_L) = \prod_{i=1}^{L} q_\lambda(x_i)\ P_\lambda(1, \ldots, 1)$$

Construct operator $\mathcal{A}_i$:

$$\begin{aligned}\mathcal{A}_i P_\lambda(x_1, \ldots, x_i, 1, \ldots, 1) &= q_\lambda(x_i) P_\lambda(x_1, \ldots, x_{i-1}, 1, \ldots, 1) \\ &= \left. \left(Q_z P_\lambda(x_1, \ldots, x_L)\right)\right|_{x_i, \ldots, x_L = 1} \Big|_{z = x_i}\end{aligned}$$

Separating operator is a chain of $\mathcal{A}_i$

$$\mathcal{S} = \mathcal{A}_1 \ldots \mathcal{A}_L$$

Q-operator method

Can also construct

$$\mathcal{S}_k = \mathcal{A}_1 \dots \mathcal{A}_k$$

which acts as

$$\mathcal{S}_k P_\lambda(x_1, \dots, x_k, 1, \dots, 1) = \prod_{i=1}^{k} q_\lambda(x_i) \; P_\lambda(1, \dots, 1)$$

Symplectic character

Let P_λ be the Symplectic character χ_λ.

$$\chi_\lambda(x_1, \ldots, x_L) = \frac{a_\mu(x_1, \ldots, x_L)}{a_\delta(x_1, \ldots, x_L)},$$
$$a_\mu(x_1, \ldots, x_L) = \det \left[x_i^{\mu_j} - x_i^{-\mu_j} \right]$$

$$\delta = (\ldots, 3, 2, 1), \qquad \mu = \lambda + \delta$$

Symplectic character

'Restricted polynomial'

$$\chi_\lambda(x_1,\ldots,x_k,1,\ldots,1) = \frac{a_\mu^{(k+1)}(x_1,\ldots,x_k)}{a_\delta^{(k+1)}(x_1,\ldots,x_k)},$$

$$a_\mu^{(k+1)}(x_1,\ldots,x_k) = \det \begin{bmatrix} \left[x_i^{\mu_j} - x_i^{-\mu_j}\right]_{i\le k} \\ \left[\mu_j^{2(L-i)+1}\right]_{i\ge k+1} \end{bmatrix}.$$

$a_\mu^{(k)}$ is proportional to a_μ;

$$\lim_{x_k,\ldots,x_L\to 1} a_\mu(x_1,\ldots,x_L) = c_k a_\mu^{(k)}(x_1,\ldots,x_{k-1})$$

Hamiltonian

The Hamiltonian

$$H_j \chi_\lambda(x_1, \ldots, x_L) = h_j(\lambda) \chi_\lambda(x_1, \ldots, x_L)$$

is of the form

$$H_j = \frac{1}{a_\delta(x_1, \ldots, x_L)} \Big[\quad \ldots \quad \Big]_j a_\delta(x_1, \ldots, x_L)$$

$[\ldots]_j$ can be found by noting

$$\left(x_i \frac{\partial}{\partial x_i} \right)^2 (x_i^{\mu_j} - x_i^{-\mu_j}) = \mu_j^2 (x_i^{\mu_j} - x_i^{-\mu_j})$$

Hamiltonian

The Hamiltonian

$$H_j \chi_\lambda(x_1, \ldots, x_L) = h_j(\lambda) \chi_\lambda(x_1, \ldots, x_L)$$

is of the form

$$H_j = \frac{1}{a_\delta(x_1, \ldots, x_L)} \Big[\quad \cdots \quad \Big]_j a_\delta(x_1, \ldots, x_L)$$

Solution:

$$[\ldots]_j = e_j \left(\left(x_1 \frac{\partial}{\partial x_1} \right)^2, \ldots, \left(x_L \frac{\partial}{\partial x_L} \right)^2 \right)$$
$$h_j(\lambda) = e_j \left(\mu_1^2, \ldots, \mu_L^2 \right)$$

Q-operator

$$\boxed{Q_z \chi_\lambda(x_1, \ldots, x_L) = q_\lambda(z) \chi_\lambda(x_1, \ldots, x_L)}$$

The eigenvalue q_λ is related to χ_λ

$$q_\lambda(z) = \frac{\chi_\lambda(z, 1 \ldots, 1)}{\chi_\lambda(1, \ldots, 1)}$$

$$= \frac{a_\delta^{(1)}}{a_\delta^{(2)}(z)} \frac{a_\mu^{(2)}(z)}{a_\mu^{(1)}}$$

Q-operator

$$\boxed{Q_z \chi_\lambda(x_1, \ldots, x_L) = q_\lambda(z) \chi_\lambda(x_1, \ldots, x_L)}$$

Solution:

$$(Q_z f)(\underset{\sim}{\mathbf{x}}) = \frac{1}{a_\delta(\underset{\sim}{\mathbf{x}})} \int_1^z \frac{\mathrm{d}\, w}{w} \int_{\mathcal{D}} \mathrm{d}\, \mathbf{Y}\ a_\delta(\underset{\sim}{\mathbf{y}}) f(\underset{\sim}{\mathbf{y}})$$

The proof is equivalent to the proof of

$$\int_1^z \frac{\mathrm{d}\, w}{w} \int_{\mathcal{D}} \mathrm{d}\, \mathbf{Y}\ a_\mu(\underset{\sim}{\mathbf{y}}) = q_\lambda(z) a_\mu(\underset{\sim}{\mathbf{x}})$$

Double integrals: $\displaystyle \int \frac{\mathrm{d}\, t_i}{t_i} \int \frac{\mathrm{d}\, y_i}{y_i}, \qquad x_i \leq \frac{y_i}{t_i} \leq x_{i+1}$

Operator $\mathcal{A}_k$

$$\boxed{\mathcal{A}_k \chi_\lambda(x_1, \ldots, x_k, 1, \ldots, 1) = q_\lambda(x_k) \chi_\lambda(x_1, \ldots, x_{k-1}, 1, \ldots, 1)}$$

Solution:

$$(A_k f)(x_1, \ldots, x_k) = \frac{1}{a_\delta^{(k)}(x_1, \ldots, x_{k-1})} \int_1^{x_k} \frac{\mathrm{d}\, w}{w} \int_{\mathcal{D}'} \mathrm{d}\, \mathbf{Y}' \; a_\delta^{(k+1)}(y_1, \ldots, y_k) f(y_1, \ldots, y_k)$$

Separating operator

Can now construct the separating operator

$$\mathcal{S} = \mathcal{A}_1 \dots \mathcal{A}_L,$$

as well as

$$\mathcal{S}_k = \mathcal{A}_1 \dots \mathcal{A}_k$$

which acts on restricted χ_λ:

$$\mathcal{S}_k \chi_\lambda(x_1, \dots, x_k, 1, \dots, 1) = \prod_{i=1}^{k} q_\lambda(x_i) \chi_\lambda(1, \dots, 1).$$

Inverse separating operator

$$\boxed{\mathcal{S}^{-1} \prod_{i=1}^{L} q_\lambda(x_i) = \frac{\chi_\lambda(x_1, \ldots, x_L)}{\chi_\lambda(1, \ldots, 1)}}$$

Solution:

$$\mathcal{S}^{-1} = \frac{a_\delta^{(1)}}{a_\delta(x_1, \ldots, x_L)} \ \det\left[\left(x_i \frac{\partial}{\partial x_i}\right)^{2(L-j)}\right] \ \prod_{i=1}^{L} \frac{a_\delta^{(2)}(x_i)}{a_\delta^{(1)}}$$

The proof of this is equivalent to

$$\det\left[\left(x_i \frac{\partial}{\partial x_i}\right)^{2(L-j)}\right] \ \prod_{i=1}^{L} \frac{a_\mu^{(2)}(x_i)}{a_\mu^{(1)}} = \frac{a_\mu(x_1, \ldots, x_L)}{a_\mu^{(1)}}$$

Inverse separating operator

$$\boxed{\mathcal{S}_k^{-1} \prod_{i=1}^{k} q_\lambda(x_i) = \frac{\chi_\lambda(x_1, \ldots, x_k, 1 \ldots, 1)}{\chi_\lambda(1, \ldots, 1)}}$$

Solution:

$$\mathcal{S}_k^{-1} = \frac{a_\delta^{(1)}}{a_\delta(x_1, \ldots, x_k)} \ \det_k \left[\left(x_i \frac{\partial}{\partial x_i} \right)^{2(k-j)} \right] \ \prod_{i=1}^{k} \frac{a_\delta^{(2)}(x_i)}{a_\delta^{(1)}}$$

Factorised Hamiltonian

It can be shown that q_λ satisfies the D.E.

$$W_x q_\lambda(x) = 0,$$

$$W_x = \prod_{n=1}^{L} \left(x\frac{\partial}{\partial x} \ + \ L\left(\frac{x+1}{x-1}\right) - \frac{2x}{x^2-1}\right)^2 - \mu_n^2.$$

Define $W_{i,j} = W_{x_i} + h_j(\lambda)$, so

$$W_{i,j} q_\lambda(x_i) = h_j(\lambda) q_\lambda(x_i)$$

$$\Rightarrow \qquad W_{i,j} \prod_{n=1}^{L} q_\lambda(x_n) = h_j(\lambda) \prod_{n=1}^{L} q_\lambda(x_n) \qquad \forall i.$$

Factorised Hamiltonian

$$W_{i,j} \prod_{n=1}^{L} q_\lambda(x_n) = h_j(\lambda) \prod_{n=1}^{L} q_\lambda(x_n) \qquad \forall i$$

Therefore any linear combination of $W_{i,j}$

$$\widetilde{H}_j = \sum_i c_i W_{i,j}$$

acts like $\mathcal{S} H_j \mathcal{S}^{-1}$, ie

$$\widetilde{H}_j \prod_{n=1}^{L} q_\lambda(x_n) = h_j(\lambda) \prod_{n=1}^{L} q_\lambda(x_n).$$

$\Rightarrow$ factorised version of the spectral problem.

Conclusion

- Found Q-operator, and $\mathcal{A}_i$ operator
- From this, constructed separating operator $\mathcal{S}$ for symplectic character χ_λ and $\mathcal{S}_k$ for restricted symplectic character
- Found inverse separating operators $\mathcal{S}$ and $\mathcal{S}_k$
- Constructed factorised Hamiltonian $\widetilde{H}_j$ from D.E. for q_λ.

Outlook:

- Asymptotics of $\chi_\lambda(\zeta_1, \zeta_2, 1, \ldots, 1)$ $(L \to \infty)$ can be obtained from asymptotics of $q_\lambda(\zeta)$ by using the inverse separating operator $\mathcal{S}_2$
- Separating operator for more general Jack polynomials of type BC, or other root systems?

Thank you for your attention

arXiv:1009.2831