TILE STRUCTURES AND THE TYPE C q-DEFORMED KNIZHNIK-ZAMALODCHIKOV EQUATION

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The functional coefficients of the states in the q-deformed Knizhnik- Zamalodchikov (qKZ) equation can be represented by tile structures, and are related to each other by means of a tile removal process.

Introduction

The operators e_i are generators of the two boundary Temperley-Lieb algebra, which act on paths $|\alpha\rangle$. The operators a_i are projectors of the Hecke algebra for $0 \le i \le N$, acting on functions ψ of $x_1, \dots, x_N$.

The qKZ equation is a compatibility condition between the actions of the e_i and the a_i. It can be written as

$$e_i|\Psi\rangle = -a_i|\Psi\rangle,$$

where

$$|\Psi\rangle = \sum_\alpha \psi_\alpha(x_1, x_2, \dots, x_N)|\alpha\rangle.$$

Finding solutions of the qKZ equation involve finding the form that the ψ_α take. This task is made easier by finding factorised expressions of the functions in terms of the Hecke projectors.

LHS OF THE qKZ EQUATION

$$e_i|\Psi\rangle = \sum_\alpha \psi_\alpha(e_i|\alpha\rangle)$$

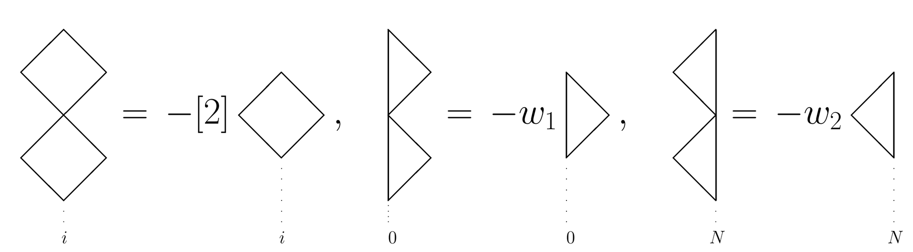
The Operators e_i

We can describe the operator e_i as a diamond tile dropped at position i (half tiles for $i = 0$ and $i = N$).

The operators satisfy the relations of the two boundary Temperley-Lieb algebra:

- The quadratic relations

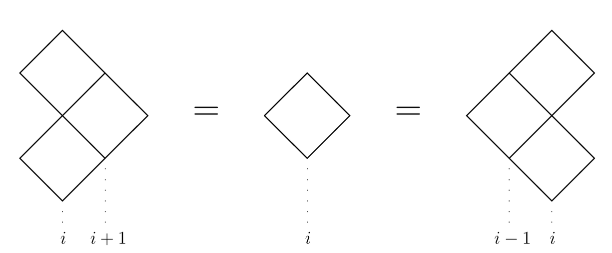
$$e_i^2 = -[2]e_i, \quad 1 \le i \le N-1$$
$$e_0^2 = -w_1 e_0,$$
$$e_N^2 = -w_2 e_N,$$



where $w_n = \frac{[\omega_n]}{[\omega_n+1]}$, for parameters $\omega_1, \omega_2 \in \mathbb{C}$.

- The Yang-Baxter relation

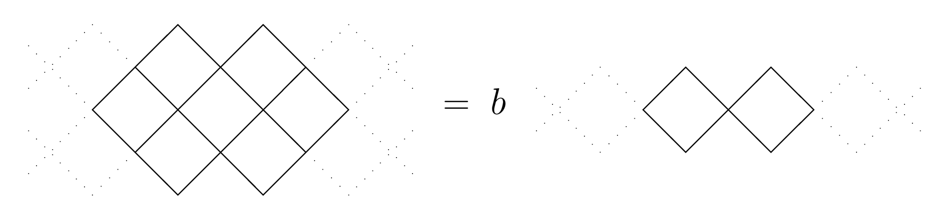
$$e_i e_{i\pm1} e_i = e_i, \quad 1 \le i \le N-1$$



- The idempotent relations

$$I_1 I_2 I_1 = bI_1, \quad I_2 I_1 I_2 = bI_2,$$

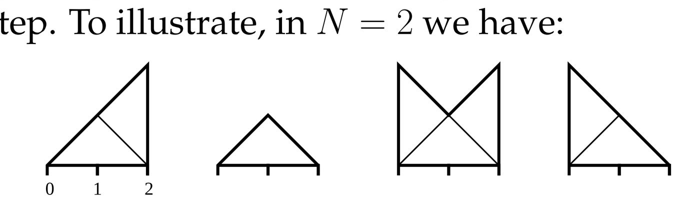
where I_1 is a combination of all e_i for i odd, and I_2 is the analogue for i even.



The effect of these relations is that there will never be a collection of tiles that is piled too high, or that has a tile suspended in mid-air.

The States $|\alpha\rangle$

The paths $|\alpha\rangle$ are independent states which have the property that, at each step, the path is either 1 unit higher or 1 unit lower than the previous step. To illustrate, in $N = 2$ we have:



The $|\alpha\rangle$ can also be expressed as a combination of e_i operators. In the $N = 2$ case above, the states are $|1\rangle = e_2e_1$, $|2\rangle = e_1$, $|3\rangle = e_0e_2e_1$, and $|4\rangle = e_0e_1$, respectively.

The properties of e_i mean that the action of one operator can dramatically change the shape of the path.

RHS OF THE qKZ EQUATION

$$-a_i|\Psi\rangle = -\sum_\alpha (a_i\psi_\alpha)|\alpha\rangle$$

The Operators a_i

The a_i satisfy the relations of the Hecke algebra of Type C:

$$a_i^2 = [2]a_i, \qquad 1 \le i \le N-1$$
$$a_0^2 = w_1 a_0$$
$$a_N^2 = w_2 a_N$$

$$a_i a_{i-1} a_i - a_i = a_{i-1} a_i a_{i-1} - a_{i-1}, \qquad 1 < i \le N-1$$
$$a_0 a_1 a_0 a_1 - a_0 a_1 = a_1 a_0 a_1 a_0 - a_1 a_0$$
$$a_N a_{N-1} a_N a_{N-1} - a_N a_{N-1} = a_{N-1} a_N a_{N-1} a_N - a_{N-1} a_N.$$

There is also a set of operators s_i which satisfy the same relations, and are defined by

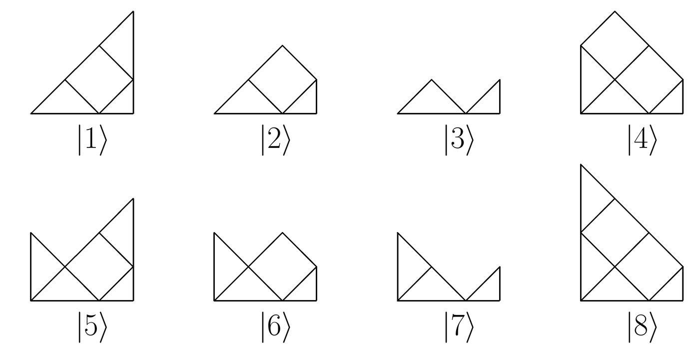
$$s_i = [2] - a_i, \qquad 1 \le i \le N-1$$
$$s_0 = w_1 - a_0$$
$$s_N = w_2 - a_N.$$

The Functions ψ_α

The action of the a_i (and the s_i) on the ψ_α is dictated by the qKZ equation and the relations described above. It is useful to set up a tiling representation for the functions identical to the corresponding $|\alpha\rangle$ paths.

SPECIFIC CASE: $N = 3$

In the case of N=3, we have 8 states:



From the qKZ equation, we get a set of relations which the ψ_α satisfy. As an example, we choose $i = 1$ and have

$$\sum_\alpha \psi_\alpha(e_1|\alpha\rangle) = -\sum_\alpha (a_1\psi_\alpha)|\alpha\rangle.$$

Considering the coefficients of $|3\rangle$, we get

$$-a_1\psi_3|3\rangle = \psi_1 e_1|1\rangle + \psi_2 e_1|2\rangle + \psi_3 e_1|3\rangle + \psi_5 e_1|5\rangle + \psi_7 e_1|7\rangle$$
$$\Rightarrow \qquad s_1\psi_3 = -w_2\psi_1 + \psi_2 + b\psi_5 + \psi_7.$$

The relations found using this method make it possible to write the functions in terms of each other.

Desired Form

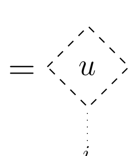
In [1], the Type A and Type B systems were each found to have maximal paths, labelled Ω. Each ψ_α is able to be expressed as a combination of operators $h_i(u)$ acting on ψ_Ω. These operators correspond to the tiles that need to be removed from the picture of ψ_Ω in order to get ψ_α.

For Type B, $h_i(u)$ is defined by

$$h_0(u) = s_0 - \frac{[\lfloor \frac{u}{2} \rfloor][\omega_1 + \lfloor \frac{u+1}{2} \rfloor]}{[u][\omega_1 + 1]}, \quad h_i(u) = s_i - \frac{[u-1]}{[u]}, \qquad 0 < i < N,$$

where u is an integer, taking the value of 1 if the tile is in a minima of the path, and increasing by 1 for each next tile up. If there are two possible values for u, it takes the larger value.

We represent $h_i(u)$ pictorially by

$h_i(u) =$ 

In the Type C case, there is no maximal path. There are, however, paths which act like maximal paths for part of the system. These are the ones with one minimum and no maxima in the bulk. For $N = 3$, these paths are $\alpha = 5$ and 7.

The desired form for each ψ_α in the Type C system is therefore $\psi_\alpha = h_i(u_1)h_j(u_2)\dots\psi_\beta$, where β is one of the psuedo-maximal states, and

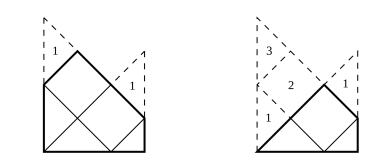
$$h_N(u) = s_N - \frac{[\lfloor \frac{u}{2} \rfloor][\omega_2 + \lfloor \frac{u+1}{2} \rfloor]}{[u][\omega_2 + 1]}.$$

To get an idea of what these factors should look like, we note that

$$h_i(1) = s_i, \quad h_i(2) = s_i - \frac{1}{[2]}, \quad h_0(3) = s_0 - \frac{[2] - w_1}{[2]^2 - 1}, \quad \text{etc.}$$

Illustrations of this for $\beta = 7$ follow in the next panel. It is hoped, but not expected, that all of the ψ_α will follow this form.

Removing Tiles



It is not complicated to express ψ_8, ψ_3 and ψ_4, as well as ψ_6 and ψ_2 in terms of ψ_7 in the desired form. However it is not so straightforward to find ψ_5 and ψ_1 in this form.

Finding expressions for ψ_5 in terms of ψ_7 is not too hard, however most of them are far more complicated than we would like. For instance,

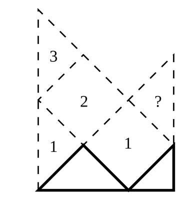
$$\psi_5 = \frac{w_1}{b(b[2] - w_1)}\left(s_1 - \frac{b[2]^2 - 1 + w_2[2](1 - w_1[2])}{[2](b - w_1w_2)}\right)\left(s_0 - \frac{[2](b - w_1w_2)}{(1 - w_2[2])}\right)$$
$$s_2\left(s_1 - \frac{1}{[2]}\right)\left(s_0 - \frac{w_1 - [2]}{1 - [2]^2}\right)\left(s_3 - \frac{b}{w_1}\right)\psi_7.$$

Some of these factors look very strange. In order to find out more about the unusual factors, we look at when they first appear.

Looking at ψ_2 in terms of ψ_7, we can see that it follows the prescription:

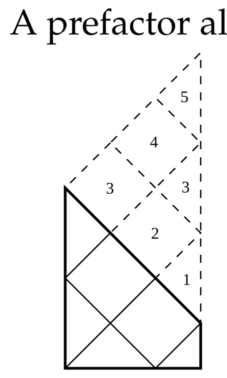
$$\psi_2 = s_0\left(s_1 - \frac{1}{[2]}\right)\left(s_0 - \frac{[2] - w_1}{[2]^2 - 1}\right)s_3\psi_7.$$

Taking one more tile to form ψ_3, however,



$$\psi_3 = \frac{w_1 + [2](b - w_1w_2)}{b[2](b - w_1w_2)}(s_2)\left(s_3 - \frac{b}{w_1 + [2](b - w_1w_2)}\right)$$
$$(s_0)\left(s_1 - \frac{1}{[2]}\right)\left(s_0 - \frac{[2] - w_1}{[2]^2 - 1}\right)\psi_7.$$

A prefactor also appears when finding ψ_8 from ψ_1:



$$\psi_8 = \frac{1}{b^2}\left(\frac{b}{(b - w_1w_2 + w_2[2] - 1)}\right)(s_3)\left(s_2 - \frac{1}{[2]}\right)$$
$$\left(s_1 - \frac{[2]}{[2]^2 - 1}\right)\left(s_3 - \frac{[2] - w_1}{[2]^2 - 1}\right)$$
$$\left(s_2 - \frac{[2]^2 - 1}{[2]([2]^2 - 2)}\right)\left(s_3 - \frac{[2]([2]^2 - w_2[2] - 1)}{[2]^4 - 3[2]^2 + 1}\right)\psi_1$$

It is expected that investigation of these two cases will provide a better understanding of the form these factors take.

CONCLUSIONS

The Type C qKZ equation coefficients begin to follow the same pattern as in Types A and B, but the relations soon prove to be more complicated. Further investigation should reveal the rules that the coefficients follow, and ultimately lead to a formulation for the general N case, which will provide a rule for finding any coefficient given one pseudo-maximal state coefficient.

References

[1] J. de Gier and P. Pyatov, *Factorised Solutions of Temperley-Lieb qKZ Equations on a Segment*, arXiv:math-ph/0710.5362, (2007).

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